When it pays off to pay tax: Growth induced by lossy redistribution

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Abstract

We demonstrate by mathematical analysis and systematic computer simulations that taxation and redistribution of wealth can lead to sustainable growth of wealth in a society. The wealth dynamics of each agent is described by a stochastic multiplicative process which, in the long run, leads to the destruction of individual wealth and the extinction of the society. When this lossy process is combined with a taxation mechanism, where some proportion of wealth is collected by a government, which further reduces a fraction as costs for administration. The remaining public good is equally redistributed to all agents. We derived conditions for under which the destruction of wealth can be turned into sustainable growth, despite the losses from the random growth process and despite the administrative costs.

The findings are verified for three different tax schemes: proportional tax, taking proportional more from the rich, and proportionally less from the poor, respectively. We discuss which of these tax schemes is optimal with respect to maximize growth of wealth under a fixed rate of administrative costs, or with respect to maximize the governmental income. This leads us to some general conclusions about governmental decisions, the relation to public good games, and the use of taxation in a risk taking society.

Introduction

Consider a society where each agent runs a similar risky business. Gains and losses are assumed to be random but proportional to the current size of the business, i.e. wealthy agents could gain more but might also loose more. After fixed time intervals, agents have to transfer a fraction of their wealth to a centralized agency, i.e. they pay taxes to a government. This authority deduces a fixed ratio of the collected wealth to compensate for administrative costs, which can be seen as a government income. The remainder is redistributed to all agents according to a given scheme such that each agent benefits from the redistribution in the same manner. Consequently, the dynamics is comprised of two different, but interlinked processes: (i) the random growth and decay of individual wealth, (ii) the collection and redistribution of taxes.

As we will see, the effect of combining proportional stochastic growth and redistribution is non-trivial. In fact, there are situations in which growth without redistribution leads to final destruction of the total wealth. When redistribution is added the situation can be turned to sustainable growth for all individuals in the long run. Surprisingly, this can even work under the presence of administrative costs, which implies that the total wealth of the society is always reduced after redistribution. Thus, each of the two processes alone cannot prevent the destruction of all wealth in the society, only the combination of the allows survival. Hence, under certain conditions it really makes sense to pay taxes, i.e. a certain loss of wealth at a given time ensures the growth of wealth in the future. This “magic” effect of induction of growth from two lossy processes is based on the portfolio effect known from investment science [1]: Gains and losses are re-balanced by a redistribution mechanism, which ensures total growth even under conditions of individual failure.

The effect has been discussed before under different names, such as repeated Kelly games, Kelly optimal portfolio, and re-balancing of asset allocations, and seems to be rediscovered from time to time in a new context [2–10].We will call the phenomenon *portfolio re-balancing effect* in the following. This interprets each agent as an asset and the society as the portfolio. Asset values change stochastically, and taxation and redistribution re-balance the values of the different assets.

The purpose of this paper is to discuss the role of the portfolio re-balancing effect for the growth of the society's wealth. By systematic computer simulation we (a) quantified conditions of the stochastic growth process, and the taxation schemes that prevent the destruction of the total wealth of the society, (b) quantified optimal tax rates which maximize growth of society's wealth, (c) quantified how a selfish government optimizes its income.

Methods

We analyzed the model by systematic computer simulations based on theoretical quantification of interesting independent parameter conditions and a theoretical analysis of its boundary cases. To describe our method we

- specify the outlined model in terms of independent random multiplicative growth processes combined with linear redistribution between the individual processes with the following independent external variables: tax rate a, rate of administrative cost b, taxation scheme, the distribution of random growth factors, and the number of agents N;
- demonstrate theoretically what conditions are most interesting to study the effect of the portfolio re-balancing effect on growth;
- demonstrate theoretically the behavior for border cases;
- give some example runs of the process; and
- describe the setup of our systematic simulation, the range of the independent variables and the extraction of the dependent variable, namely the average growth factor.

Model specification

Let us consider a system of N agents, each of which is characterized by a positive scalar value $x_i(t)$. This shall denote its *wealth* at time t, with $x(t) \in \mathbb{R}^n$ being the *wealth vector* of all $x_i(t)$. Let us denote the *total wealth* at time t by $X(t) = \sum_{i=1}^{N} x_i(t)$.

Given an initial wealth vector $x(0)$ the dynamical equation is

$$x(t+1) = \text{redistribution}(\text{production}(x(t))) \tag{1}$$

with redistribution and production being self-maps on the space of wealth vectors $\mathbb{R}^n$ which we specify as follows.

Redistribution We quantify the redistribution function with three different taxation schemes: proportional taxation, a "dynamic maximum"-scheme where agents have to pay everything above a (dynamically chosen) maximal tax-free income, and a "dynamic fee"-scheme where agents have to pay either a (dynamically chosen) fee (like a per capita premium) or all their wealth if they cannot afford the full fee (no worries, agents get back some income because of the redistribution). For all three schemes we specify the same two independent parameters: the *tax rate a* which determines the fraction of total wealth withdrawn from the agents for redistribution and government income, and the *rate of administrative cost*, b, which determines the fraction of the total taxes lost for redistribution, the fraction also determines the *government income.*

Let us call the amount of taxes collected from agent i to be $\text{tax}_i(x)$. Notice that it depends on a full wealth vector. This enables to define dynamically adjusted taxation schemes which take the distribution of wealth into account. Naturally, $\text{tax}_i(x)$ should be a number between zero (no tax) and x_i (tax equals wealth).

The total amount of taxes raised is collected by a government at a central place, which involves administrative costs, which are assumed to be proportional to the amount of taxes raised, i.e. $b \in [0, 1]$ denotes the *rate of administrative cost.* Consequently, the *public good* for redistribution is the raised taxes minus the cost:

$$\text{pg}(x) = (1 - b) \sum_{i=1}^{N} \text{tax}_i(x), \tag{2}$$

while the *government income* is

$$\text{pg}(x) = b \sum_{i=1}^{N} \text{tax}_i(x). \tag{3}$$

The public good is redistributed equally among all agents, i.e. for every agent wealth increases by an amount $\text{pg}(x)/N$. The redistribution function is thus

$$\text{redistribution}_i(x) = x_i - \text{tax}_i(x) + \frac{\text{pg}(x)}{N} \tag{4}$$

In order to realize the three tax regime we have to specify the tax function $\text{tax}_i(x)$. All of these involve a parameter $a \in [0, 1]$ which represents the *tax rate*, i.e. the fraction of wealth which is collected from the total wealth

$$\sum_i \text{tax}_i(x) = a \sum_i x_i = aX. \tag{5}$$

The taxation schemes differ in how much is taken from whom, to reach the fraction a of the total wealth.

(i) *Proportional tax* is the classical taxation scheme where each agent has to pay the a fraction a of its individual wealth

$$\text{tax}_i(x) = ax_i. \tag{6}$$

(ii) *Dynamic fee* charges a fixed fee $c_{\text{fee}}(x) > 0$ from everyone if possible, otherwise all wealth is charged.

$$\text{tax}_i(x) = \min\{x_i, c_{\text{fee}}(x)\} \tag{7}$$

In the latter case, the agent still receives its proportion from the public good, so it will not be without wealth after taxation and redistribution. The fee has to be such that (5) is fulfilled for the given wealth vector. (It is easy to see that this is possible and unique.)

(iii) *Dynamic maximum* charges all wealth exceeding a threshold of tax free income $c_{\max}(x) > 0$ from everyone. Every agent with wealth below $c_{\max}(x) > 0$ pays no taxes.

$$\text{tax}_i(x) = \max\{x_i - c_{\max}(x), 0\} \tag{8}$$

The threshold has to be determined such that (5) is met. Notice, that the wealth of an agent who has to pay taxes is larger than $c_{\max}(x)$ after redistribution because of its share from the public good.

The three schemes above are comparable in that the total amount of raised taxes is a always a fraction of a of the total wealth, regardless of the shape of the distribution of wealth. Thus, they all deliver a public good of $\text{pg}(x) = (1 - b)aX$.

The schemes differ in from whom the tax is raised: (i) proportionally from everyone, (ii) more than proportionally from the poor, or (iii) more than proportionally from the rich. More specifically, dynamic fee (ii) and dynamic maximum (iii) represent extreme cases of regressive and progressive taxation systems. In progressive and regressive taxation schemes tax rates differ in given wealth brackets. In our case, two brackets are divided by the dynamic border: $[0, c(x)]$ and $[c(x), \infty]$. The dynamic fee scheme (where

$c = c_{\mathrm{fee}}$) is regressive with tax rate 100% in the lower tax bracket and 0% in the upper. The dynamic maximum scheme (where $c = c_{\max}$) is progressive with tax rate 100% in the upper tax bracket and 0% in the lower. Figure 1 demonstrates for an example of six agents with different wealth what tax will be charged from each agent and how wealth looks after redistribution for each of the three schemes.

Note, that the dynamic fee $c_{\mathrm{fee}}(x)$ and the dynamic maximum $c_{\max}(x)$ are implicitly defined, to meet the condition of Eq. (5). In realistic taxation systems, it might seem impractical to determine the fee and the maximum after the current wealth of all agents is known. In reality one would only adjust thresholds for the next turn. But this would cause only a small delay, which likely does not affect the core results presented here (a point left for future discussion).

Production The production of wealth is assumed to be based an individual multiplicative stochastic growth event:

$$\mathrm{prod}_i(x_i) = \eta_i(t)x_i \tag{9}$$

where $\eta_i(t)$ is a realization of the positive random variable η. If agent i at time t has wealth $x_i(t)$ then after production its wealth is $\eta_i(t)x_i(t)$. When $\eta_i(t) < 1$ the wealth declines, otherwise it grows. Without subscripts i, prod and $\eta(t)$ are meant as vectors. Thus, the growth dynamics for the wealth vector x reads $\mathrm{prod}(x) = \eta(t)x$. The product $\eta(t)x$ is meant component-wise, $\eta(t)$ being an equally sized vector of independent realizations of η.

On the distribution of random growth factors

The portfolio re-balancing effect can have drastic effects by making the difference between sustainable growth and extinction. In the following we quantify the conditions for this situation. Consequently, we fix the log-normal distribution as the appropriate distribution for simulation and furher state a reasonable condition which determines its two parameters.

To that end, we discuss production without redistribution, i.e. $a = 0$ in (1). As production does not involve interaction it is enough to focus on a single agent. Let η have finite variance. With $x(0) = 1$ it holds

$$x(t+1) = \eta(t)x(t) = \prod_{s=0}^{t} \eta(s)$$

which is equivalent to

$$\log x(t+1) = \log \eta(t) + \log x(t) = \sum_{s=0}^{t} \log \eta(s). \tag{10}$$

Whether or not wealth grows in the long run depends on the properties can be quantified by looking at the mean of the random variables η and its logarithm $\log \eta$. Let us define

$$\begin{aligned} \mu_\eta &= \langle \eta \rangle\,, & \sigma_\eta^2 &= \left\langle \eta^2 \right\rangle - \langle \eta \rangle^2\,, \\ \mu_{\log \eta} &= \langle \log \eta \rangle\,, & \sigma_{\log \eta}^2 &= \left\langle (\log \eta)^2 \right\rangle - \langle \log \eta \rangle^2\,. \end{aligned} \tag{11}$$

It is known (but always surprising seeing it in simulation) that there are distributions for η where the expected value of wealth grows, while every individual wealth trajectory dies out. Elementary explanations of this effect are given in [1,10,11]. The condition for this "strange" situation is

$$\mu_{\log \eta} < 0 < \log \mu_\eta$$

which is equivalent to

$$\langle \eta \rangle_{\mathrm{geo}} = \exp \mu_{\log \eta} < 1 < \mu_\eta = \langle \eta \rangle$$

i.e. the arithmetic mean of η being larger than one, while its geometric mean is less than one.

The central limit theorem applied to (10) implies that the distribution of the random variable $\log x(t)$ gets closer and closer to a normal distribution $\mathcal{N}(\mu_t, \sigma_t)$ with

$$\mu_t = t\mu_{\log\eta}\ ; \quad \sigma_t^2 = t\sigma_{\log\eta}^2 \tag{12}$$

Thus, for $t \to \infty$ the distribution of $x(t)$ approaches the log-normal distribution $\log\text{-}\mathcal{N}(t\mu_{\log\eta}, \sqrt{t}\sigma_{\log\eta})$. The time-dependent mean of the distribution $\log\text{-}\mathcal{N}(\mu_t, \sigma_t)$ is

$$\exp\left(\mu_t + \frac{\sigma_t^2}{2}\right) = \exp\left(t\mu_{\log\eta} + t\frac{\sigma_{\log\eta}^2}{2}\right) = \left(\exp\left(\mu_{\log\eta} + \frac{\sigma_{\log\eta}^2}{2}\right)\right)^t$$

Thus, it is well possible that $\mu_{\log\eta}$ is negative while at the same time the mean of $\log\text{-}\mathcal{N}(\mu_t, \sigma_t)$ is larger than one and grows with t, when $\sigma_{\log\eta}^2 > -2\mu_{\log\eta}$. If $\exp(\mu_{\log\eta} + \sigma_{\log\eta}^2/2) > 1$ the mean of $x(t)$ diverges towards $+\infty$ although the die-out probability of an individual trajectory is one. It can be shown that for long time any single trajectory of $x(t)$ grows only with the geometric mean $\langle\eta\rangle_{\text{geo}}$.[1]

This interesting, but seemingly contradictory, conclusion forms the basis for the effect discussed in the following: growth induced by coupling lossy multiplicative stochastic growth and lossy redistribution. Redistribution among different independent processes helps the system to realize a growth rate somewhere in between the geometric and the arithmetic mean of η.

Based on these consideration we have chosen the lognormal distribution as the distribution of η. For simulation purposes we also aimed at reducing the number of independent parameters and chose that only lognormal distributions are of interest where $\langle\eta\rangle \cdot \langle\eta\rangle_{\text{geo}} = 1$. Thus, the two e of the lognormal distibution μ and σ (as mean and standard deviation of the underlying normal distribution) are represented by one free parameter which allows for different skewness, but keeps the balance of the expected ($\langle\eta\rangle$) an the realized ($\langle\eta\rangle_{\text{geo}}$) growth rate at one. This enables that destruction and sustainable growth are theoretically possible.

Theoretical analysis of border cases and an example

We are interested in the average realized growth rate and its dependence on the independent parameters. For some border cases we theoretically derive that growth is exponential $X(t+1) = gX(t)$ and also quantify the magnitude of the growth rate g.

Only redistribution, no production ($\langle\eta\rangle_{\text{geo}} = 1 = \langle eta\rangle$): for any taxation scheme, and any N it holds $g = (1 - ab)$. Thus, there is never growth.

Only production no redistribution ($a = 0$): for any taxation scheme, any b and any N it holds $g = \langle\eta\rangle_{\text{geo}}$.

100% tax and infinite number of agents ($a = 1$, $N = \infty$): All trajectories act as one, and for any taxation scheme $g = (1 - ab)\langle\eta\rangle$. The growth with the mean can be realized because in an infinite society

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[1] Similar considerations can be obtained from the probability density function (pdf) $p(x,t)$, which represents the "theoretic" histogram of several random trajectories of x at time t. For large t, $p(x,t)$ is close to the log-normal distribution

$$p(x,t) = \frac{1}{x\sigma_{\log\eta}\sqrt{t}\sqrt{2\pi}} \exp\left[-\frac{(\log(x) - t\mu_{\log\eta})^2}{2t\sigma_{\log\eta}^2}\right] \tag{13}$$

The corresponding cumulative distribution function (cdf) is

$$P(x,t) = \frac{1}{2} + \frac{1}{2}\text{erf}\left[\frac{\log(x) - t\mu_{\log\eta}}{\sigma_{\log\eta}\sqrt{t}\sqrt{2}}\right]. \tag{14}$$

It is easy to see that $P(x,t) \overset{t\to\infty}{\longrightarrow} 1$ for any $x > 0$ when $\mu_{\log\eta} < 0$. Specifically this means that any q-Quantile of the distribution of $x(t)$ goes to zero with rising t. This implies that for an $x^* > 0$ (but arbitrarily close to zero) it holds $\Pr(x(t) < x^*) \overset{t\to\infty}{\longrightarrow} 1$. Thus, the probability of extinction of a trajectory is one, and it is well possible that $\langle x(t)\rangle \overset{t\to\infty}{\longrightarrow} +\infty$.

no rare but large growth event is “missing”. This happens in finite societies and is the reason that in practice only lower growth rates realize.

Based on the last two cases we argue, that for intermediate a and finite N the average growth rate lies somewhere in between.

Figure 2 gives an example, where we fixed $\langle\eta\rangle = 1.5$, which implies $\langle\eta\rangle_{\mathrm{geo}} = 0.667$, and conseqeuntly $\mu = -0.405$ and $\sigma = 1.274$. Under this distribution wealth declines with a probability of 62.5%, but it at least doubles with probability 19.4%, and it will be ten times larger (or more) with probability 1.7%. Tax and admin are set at intermediate levels $a = 0.3$, $b = 0.2$. Trajectories are computed trajectories according to (1) for a society of $N = 10$ agents, each starting with wealth equal to one. Trajectories are shown for all three tax schemes. Each trajectory is computed with the same realizations of the random variables $\eta_i(t)$, to allow for a direct comparison of the different taxation schemes.

In Figure 2 taxation schemes (iii) “dynamic maximum” and (i) “proportional tax” lead to the growth of total wealth whereas scheme (ii) “dynamic fee” seems to imply no growth and no decline on average. No redistribution would imply decline, as well as pure redistribution results with a growth factor of $1 - ab = 0.94$. Comparing the different schemes, we find that the total wealth grows best under scheme (iii) “dynamic maximum”, followed by (i) “proportional tax”, which performs better than (ii) “dynamic fee”. This performance ranking holds even if we vary the essential parameters admin rate b, tax rate a and the size of the population N as shown in the lower part of Fig. 2. Regarding their general impact on the dynamical behavior, we see that for a higher admin rate ($b = 0.6$) growth might turn to decline. A lower tax rate ($a = 0.01$), however can also imply destruction of wealth for all three taxation schemes. In this case the portfolio re-balancing effect of redistribution is not used well and this is not compensated by the savings from the lower loss of redistribution ($1 - ab = 0.998$). Finally, in a larger society with $N = 100$ all schemes achieve growth factors larger than one.

Simulation setup, independent and dependent variables

For each of our three taxation schemes we aimed to get an overview about the dependence of the average growth factor on the tax rate and the rate of administrative costs. Further on we wanted to control for the effect of society’s size and lognormal distributions which are more or less right-skew.

Consequently, we set up a systematic computer simulation to estimate *average growth factors* g. See Table 1 for the list of independent variables, their ranges in the simulation setup and how average growth factors were estimated. We covered the (b, a)-plane by a fine grid while the number of different sizes of the society and different distributions was kept low to make computations finish within less than a week. See also the `matlab`-code in the supporting material which produces the simulation data (see function `dataMSPgrowthrates`).

As our focus is on maximizing the growth factor, optimal tax rates, and government income let us define further variables which depend on the average growth factor as a function of a and b, $g(b, a)$: For a fixed admin rate b we define the *maximal growth factor* $g_{\max}(b) = \max_a g(b, a)$ and the *optimal tax rate* $a_{\mathrm{opt}}(b) = \mathrm{argmax}_a g(b, a)$. (Note, that argmax is not necessarily unique, but empirical results support the conjecture that there is only one local maximum, and consequently $a_{\mathrm{opt}}(b)$ is uniquely defined.) The *rate of government income* at the current total wealth is $\mathrm{gov}(b, a) = b\,a\,X(t+1)/X(t) = b\,a\,g(b, a)$. Under the assumption that a government is forced to choose the growth maximizing optimal tax rate we define the *rate of government income under optimal tax rate* as a function of b by $\mathrm{gov}_{\mathrm{opttax}}(b, a) = b\,a_{\mathrm{opt}}\,g_{\max}(b)$.

Results

Outline figures with results of systematic simulations

We visualize our simulation results exemplary for $N = 10$ and the intermediate distribution $(\langle\eta\rangle, \langle\eta\rangle_{\text{geo}}) = (1.5, 0.667)$ in Figure 3. We shows in part **A** the average growth factor $g(b,a)$ color-coded in the (b,a) parameter plane for each of the three tax schemes. In each plot, the solid line divides the "zone of sustainable growth of wealth" (yellow to red) from the "zone of wealth destruction" (yellow to blue). The dashed line shows the optimal growth maximizing tax rate $a_{\text{opt}}(b)$ for a given admin rate. Figure 3**B**, we show the critical lines dividing the zones of growth and destruction and the optimal tax rates in one plot to compare the three taxation schemes. (In all plots the black dotted line shows the maximally possible size of the zone of growth, where $(1 - ba)\langle\eta\rangle = 1$. Above this line it is trivial that wealth can not grow.

Figure 4**A** shows the *maximal growth factor* $g_{\max}(b)$, 4**A** the *optimal tax rate* $a_{\text{opt}}(b)$, and 4**C** the rate of government income at optimal tax rate $\text{gov}_{\text{opttax}}(b)$. All are functions of the admin rate b and they are shown for all three taxation schemes in our standard color-code.

Figure 5–8 are extensions of Figures 3 and 4 by showing also the data for $N = 100$, the less risky, and the more risky distribution of η. Figures 5, 6, and 8, are the pure extensions of Figures 3**A**, 3**B**, and 4, while Figure 7 is a regrouping of lines to different subplots focusing on comparison of $N = 10$ and $N = 100$.

On growth and destruction of wealth

Figure 3 and related Figures 5, 6 7 summarize what combination of tax rates and admin rate allow for society's wealth to grow sustainable. It is interesting to focus on situations where either the tax rate a or the admin rate b is fixed but the other changes. The parameter dependence of the average growth factor $g(b,a)$ is as follows:

Constant tax rate a: Raising the admin rate b always lowers the growth factor and turns the growth regime at some point into the destruction regime as one should expect.

Constant admin rate b: The average growth factor is non-monotonic in a. For very high and very low tax rates, the growth factor is the lowest and can lead to wealth destruction, while only intermediate tax rates prevent this. High tax rates tend to lower the growth factor because a larger fraction of the total wealth is reduced by the admin rate (see the definition of the public good (2)). On the other hand, very low tax rates lower the growth factor because the portfolio re-balancing effect is not used optimal, thus part of the wealth is "gambled away".

This characterization is ubiquitous for all taxation schemes, both population sizes and all three distributions of η. It holds ubiquitous that the zone of sustainable growth for "dynamic fee" is contained in the zone of growth of "proportional tax", which is further contained in the zone of growth of "dynamic maximum", when N and the distribution of η are kept constant. When the taxation scheme and the distribution of η is kept constant the zone of growth for the smaller society ($N = 10$) is always contained in the zone of grow the larger society ($N = 100$).

These findings suggest that the portfolio re-balancing effect is used more effective, when the society is large and when proportionally more is taken from the rich (as in "dynamic maximum"), than from the poor (as in "dynamic fee"). (Simple inclusions of zones of growth do not hold for comparisons of our low risk, intermediate and risky distribution of η. We refrained from comparing them in detail, because our balancing condition $\langle\eta\rangle \cdot \langle\eta\rangle_{\text{geo}}$ was ad hoc.)

On growth maximizing tax rates and taxation schemes

(1) *What is the optimal tax rate $a_{\text{opt}}(b)$ and how does it differ between the three taxation schemes?* The optimal tax rate is 100% under admin rate $b = 0$ for any tax system but it declines fast with rising admin rate as can be seen in Figure 4**A**. Within the range $0 < b < 0.35$ the "dynamic maximum" scheme reaches the lowest optimal tax rate, "dynamic fee" scheme the highest. E.g. for a realistic admin rate of 20% the optimal tax rate in the "dynamic maximum" scheme and the "proportional tax" scheme is less than 30%, but larger than 50% in the "dynamic fee" scheme. This ranking is inverted for large admin rates. From Figure 8 it can be seen that these rankings and the switch of the ranking also holds for larger societies being more drastic for low admin rates and less drastic for high admin rates. For riskier societies the optimal tax rates are in general larger.

(2) *Which taxation scheme reaches the largest average growth factor for a given admin rate and optimal choice of the tax rate?* As can be seen in the central panel of Fig. 4**B** the "dynamic maximum" scheme achieves the highest maximal growth factors for all admin rates. The "proportional tax" scheme is always second and the "dynamic fee" scheme ranks last. Hence, the largest growth factor, which maximizes the total welfare for the society as a whole, is reached with the "dynamic maximum" scheme that takes more than proportional from the rich.

On income maximizing governments

The tax rate is usually set by the government. In a democratic society the government is chosen by elections. When we focus on simple majority voting and two competing candidates or parties both try to make policy proposals for a tax rate which achieves a majority. Naturally the poor agents are for high redistribution while the rich are for low. Competitive policy proposals lie at the median of citizens preference (see [12]), if there is further an optimal tax rate which maximizes the average growth rate for all then we might assume ad hoc, that the democratic process would choose governments which set tax rates close to the optimal tax rate. This is the reason why the rate of government income under optimal tax rate $\mathrm{gov}_{\mathrm{opttax}}$ is of interest, because based on this assumption we can ask what admin rate a government might choose to maximize its income.

The rate of administrative costs is usually something which is also under the control of the government, but a democratic process of government need not force governments to find a societal optimum (no admin cost!) because all parties aiming to gain office have the same interest of a large government income.

Based on these two assumption and simulation results we can answer three questions.

(3) *Which admin rate b would a self interested government choose?*

Figure 4**C** shows that the rate government income under optimal tax rates is not monotonic in b. In particular, for higher values of b the growth of the total wealth becomes smaller, hence even a income maximizing government has no incentives to raise their admin rate to the largest possible. This is because large admin rates naturally reduce growth. The admin rate where the government income is maximal is marked by "$*$" in all three panels. It varies with the taxation scheme: lowest admin rates for "dynamic fee", highest admin rates for "dynamic maximum". This ranking only changes for riskier distribution of η (see Figure 8), with"proportional tax" having lowest admin rates.

(4) *Which taxation scheme would a self interested government choose?*

Looking at absolute values of $\mathrm{gov}_{\mathrm{opttax}}(b)$ a self interested government would choose "dynamic fee" because it gives the maximum income of all schemes, even at moderate admin rates. Thus, the largest government income is reached with a scheme that takes more than proportional from the poor. This is caused mainly due to the fact that the optimal tax rate is much larger under the "dynamic fee" scheme. Consequently, the share raised by the admin rate is also larger as under other schemes. Thus, this result crucially depends on the assumption that a government will automatically move to the optimal tax rate, due to a democratic choice, while the taxation scheme itself is not suspect to democratic decision.

(5) *Which taxation scheme delivers the largest average growth factors under a self interested government?*

Looking at the "$*$"-symbols in Figure 4**B** which come from optimal admin rates with respect to $\mathrm{gov}_{\mathrm{opttax}}(b)$ in Figure 4**C**, we find that "proprotional tax" delivers the highest average growth factors. Why does this hold although "dynamic fee" can deliver the highest income for the governemt as seen in

(4)? Because "dynamic fee" has much lower growth rates than the other schemes in general. Why does this hold although "dynamic maximum" always delivers higher growth rates than the other regimes as seen in **(2)**? Because "dynamic maximum" attracts the government to raise the admin rate to optimize its income.

Discussion

The answers to questions **(2)**, **(4)**, and **(5)** deliver three different answers about the choice for one of the three taxation schemes. **(2)** suggest that the progressive taxation scheme "dynamic maximum" should be chosen because it is always superior when a rate of administrative cost is fixed. Consequently, taking proportionally more from the rich is socially optimal when we can assume that the rate of administrative costs is an externally fixed parameter. **(4)** instead suggests, that a selfish government would decide for the progressive taxation scheme "dynamic fee" because under this scheme growth optimizing tax rates are much higher which leads to higher government income. Finally **(4)** shows that from the three taxation schemes the proportional tax achieves the highest average growth factor when an income maximizing government which can freely adjust the rate of administrative cost is assumed. The "dynamic fee" scheme turns out to be to inefficient in turning on the portfolio re-balancing effect to enhance growth, while "dynamic maximum" turns to to give incentives to raise the admin rate to such an extend that the loss due to this outweighs its efficiency in using the portfolio re-balancing effect.

By focusing on the portfolio re-balancing effect we propose a new approach to think about taxation. With our simulation, we have shown that taxation can be a crucial ingredient to ensure the survival of a society of wealth-creating agents, given that wealth production is governed by independent multiplicative stochastic processes. The latter one seems reasonable given that fact that investment success is usually proportional due to the invested amount. Further on, empirical findings about the wealth distribution in democratic societies [13] point to multiplicative growth.

On a more general level, our model supports the idea that it pays off for selfish agents to share their wealth with others, by paying taxes - provided the conditions derived above are fulfilled. Even if agents with a large gain have to pass on some of their wealth immediately and thus keep less, they profit from this in the long run, because otherwise they have to quit after their wealth is gone. Such a situation has much in common with the dilemma of cooperation. In a game theoretical setting, redistribution of a public good is the idea of the public goods game (with the prisoner's dilemma as its two player version).

In the same spirit, paying taxes can be seen as an act of cooperation, which seems irrational but ensures the sustainable growth of wealth, e.g. as an evolutionary promoted behavioral program of successful societies.

At difference to the classical public goods game, where the public good is multiplied by an efficiency factor larger than one, our model does not have such an amplification. On the contrary, from the collected public good a fraction is subtracted for administrative costs, which is equivalent to an efficiency factor less than one. Consequently, the emergence of cooperation, i.e. the sharing of wealth in order to sustain a long term growth, is even more subtle in the tax paying example than in the classical public goods game. In the public goods game, intelligent agents can "solve" the cooperation dilemma either by using the categorical imperative "If I wish others should not defects I also shouldn't" or by focusing on societal efficiency. Both of these rational arguments are less obvious to apply when administrative costs are at work.

The portfolio re-balancing effect might also be of relevance in other areas, such as biodiversity [14] or knowledge sharing, to enhance innovativeness in social and economic systems. Eventually, it resembles the proposal of some successful religions, to "give a tenth part of your wealth to the needy" – because it ensures better growth of society's wealth.

If we assume that different societies of wealth producing agents compete, evolution would promote those societies with higher overall growth factors of their total wealth. Thus, there should be an evolutionary adaptation towards the optimal tax systems without assuming other forces. Such an idea is closely related to group selection as a mechanism to promote the evolution of cooperative behavior [15]. But, as we already pointed out, the benefit of cooperation, i.e. of paying taxes, is not that obvious in our example, even for “smart” agents.

Finally we ask if we can draw conclusions for large societies of some millions as in the real world societies? We speculate that sizes larger than $N = 1000$ imply even lower optimal tax rates because the portfolio re-balancing effect works even with very low tax rates and thus admin costs can be saved just by low tax rates. But on the other hand we speculate that riskiness of individual stochastic growth also rises in larger societies which consequently implies higher optimal tax rates (see Figure 8). It is much less likely to have a big success in a large society because there are more competitors, but on the other hand really large successes are possible because there are many customers. In conclusion, we speculate that our results are probably still valid for societies of real world sizes.

Acknowledgments

References

1. Luenberger D (1998) Investment Science. Oxford University Press.
2. Kelly J J (1956) A new interpretation of information rate. Information Theory, IRE Transactions on 2: 185 -189.
3. Bouchaud J, Mézard M (2000) Wealth condensation in a simple model of economy. Physica A: Statistical Mechanics and its Applications 282: 536–545.
4. Malcai O, Biham O, Richmond P, Solomon S (2002) Theoretical analysis and simulations of the generalized lotka-volterra model. Phys Rev E 66: 031102.
5. Marsili M, Maslov S, Zhang YC (1998) Dynamical optimization theory of a diversified portfolio. Physica A: Statistical and Theoretical Physics 253: 403–418.
6. Medo M (2008) Breakdown of the mean-field approximation in a wealth distribution model. Arxiv preprint arXiv:08094139 .
7. Medo M, Pis’mak YM, Zhang YC (2008) Diversification and limited information in the kelly game. Physica A: Statistical Mechanics and its Applications 387: 6151–6158.
8. Slanina F (1999) On the possibility of optimal investment. Physica A: Statistical Mechanics and its Applications 269: 554–563.
9. Slanina F (2004) Inelastically scattering particles and wealth distribution in an open economy. Phys Rev E 69: 046102.
10. Yaari G, Solomon S (2010) Cooperation evolution in random multiplicative environments. Eur Phys J B .
11. Redner S (1990) Random multiplicative processes: An elementary tutorial. Am J Phys 58: 267.
12. Downs A (1957) An Economic Theory of Democracy. Harper & Row Publishers, New York.
13. Yakovenko VM, Rosser JB (2009) Colloquium: Statistical mechanics of money, wealth, and income. Rev Mod Phys 81: 1703–1725.

14. Schindler DE, Hilborn R, Chasco B, Boatright CP, Quinn TP, et al. (2010) Population diversity and the portfolio effect in an exploited species. Nature 465: 609–612.

15. Nowak MA (2006) Five Rules for the Evolution of Cooperation. Science 314: 1560-1563.

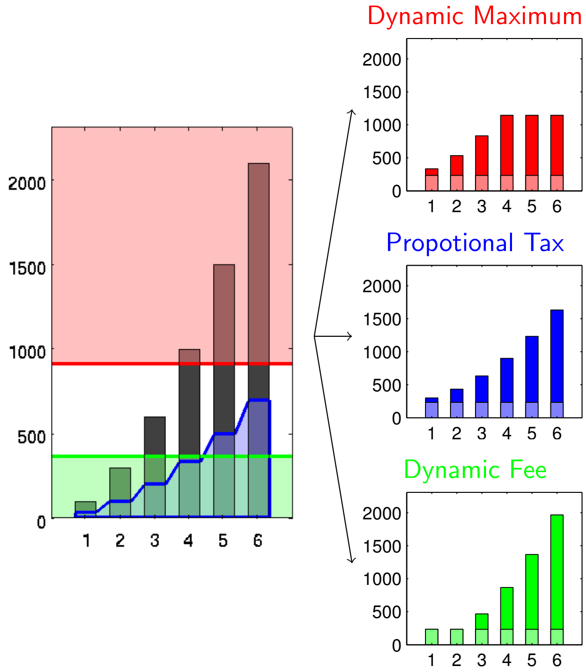

Figure 1. Demonstration of the redistribution function. *Proportional*, *dynamic fee* and *dynamic maximum* tax functions. Six agents with wealths $x = [100, 300, 600, 1000, 1500, 2100]$. All tax functions are such that $a = 1/3$ of the total wealth $(= 5600)$ is taxed, the administrative cost is set to 25% $(b = 0.25)$. In numbers:pg$(x) = 1400$, $c_{\text{fee}} = 366\frac{2}{3}$. The maximum to tax all above is $c_{\text{fee}} = 911\frac{1}{9}$.

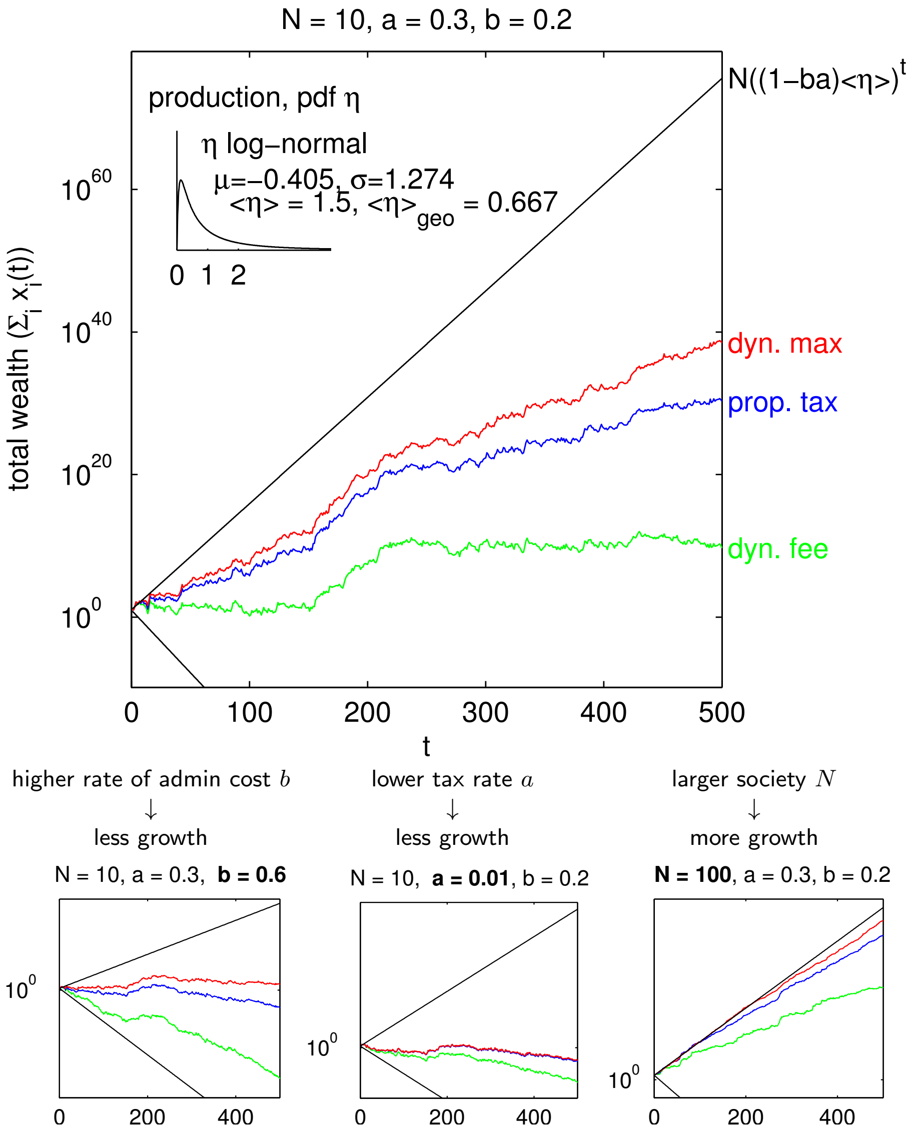

Figure 2. Example trajectories for three tax schemes according to Eq. 1 and three exemplary modifications. Trajectories computed with the same realizations of the random variables $\eta_i(t)$. Black lines show limiting cases: lower line shows “no tax” $X(t) = N(\langle\eta\rangle_{\text{geo}})^t = 10 \cdot (0.667)^t$; upper line shows “full tax and infinite number of agents” $X(t) = N((1 - ab)\,\langle\eta\rangle) = 10 \cdot (1.41)^t$.

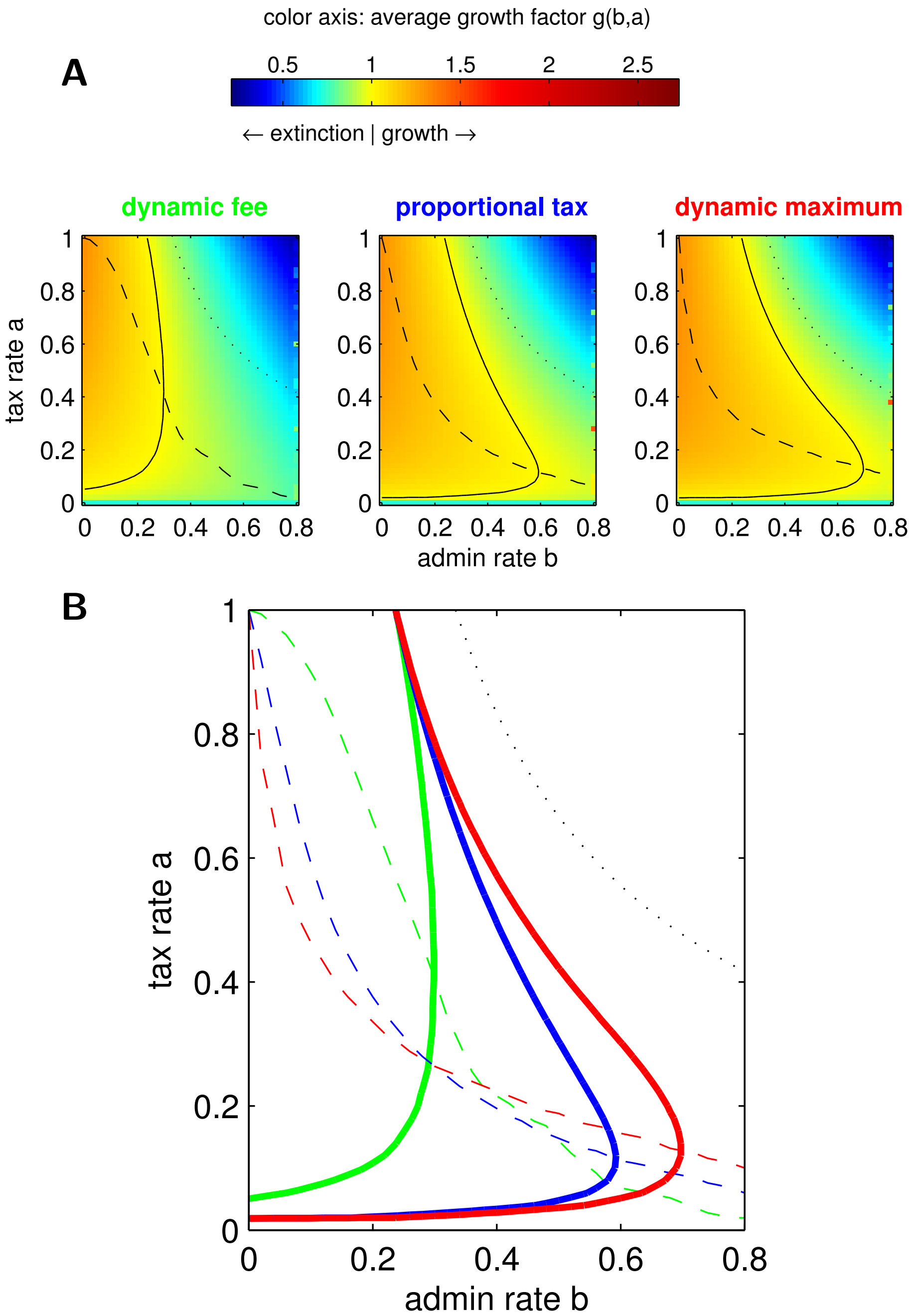

Figure 3. Average growth factor $g(b,a)$ for $N = 10$, $\langle\eta\rangle = 1.5$, and $\langle\eta\rangle_{\text{geo}} = 0.667$ (cf. Figure 2). **A** $g(b,a)$ in the (b,a) parameter plane color-coded as specified in color bar for all three taxation schemes. Solid lines divide zones of wealth growth from wealth destruction. Dashed lines are optimal tax rates for given admin rate a_{opt}. Above dotted line wealth destruction is trivial. **B** Lines of A in one plot to compare. Colors indicate tax schemes. Linestyles as above.

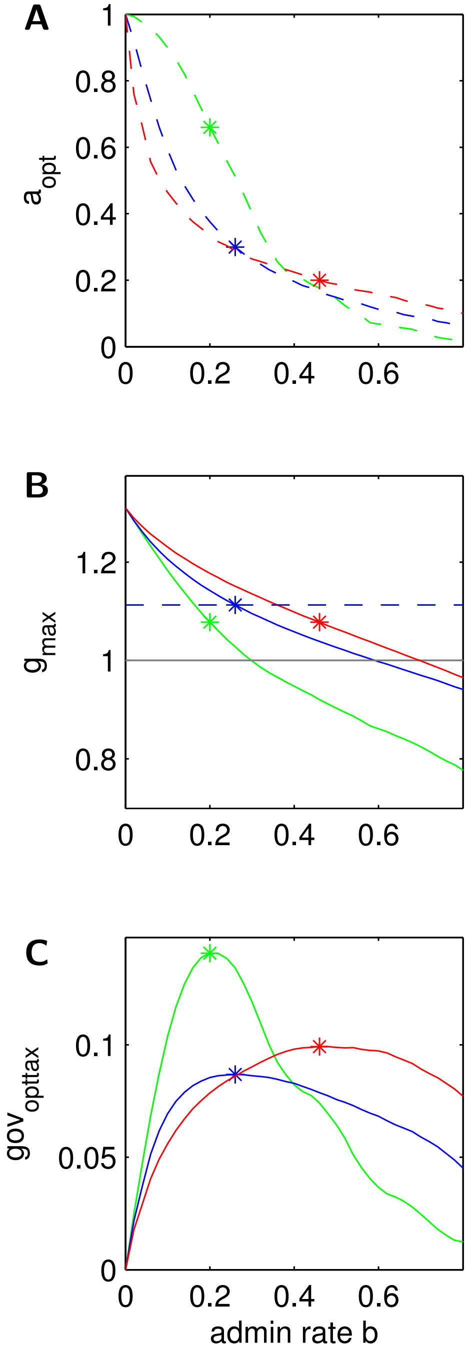

Figure 4. **A** Optimal tax rate $a_{\mathrm{opt}}(b)$, **B** maximum achievable growth rate $g_{\max}(b)$, **C** governmental income rate under optimal tax rate $gov_{\mathrm{opttax}}(b)$ for different tax schemes: (i) proportional tax (blue), (ii) dynamic fee (green), (iii) dynamic maximum (red). Stars indicate the location of the maximum of $\mathrm{gov}_{\mathrm{opttax}}$ in all plots. The dashed line is to show that under the maximal $\mathrm{gov}_{\mathrm{opttax}}$ proportional taxation gives the largest growth factor. Parameters as in 3.

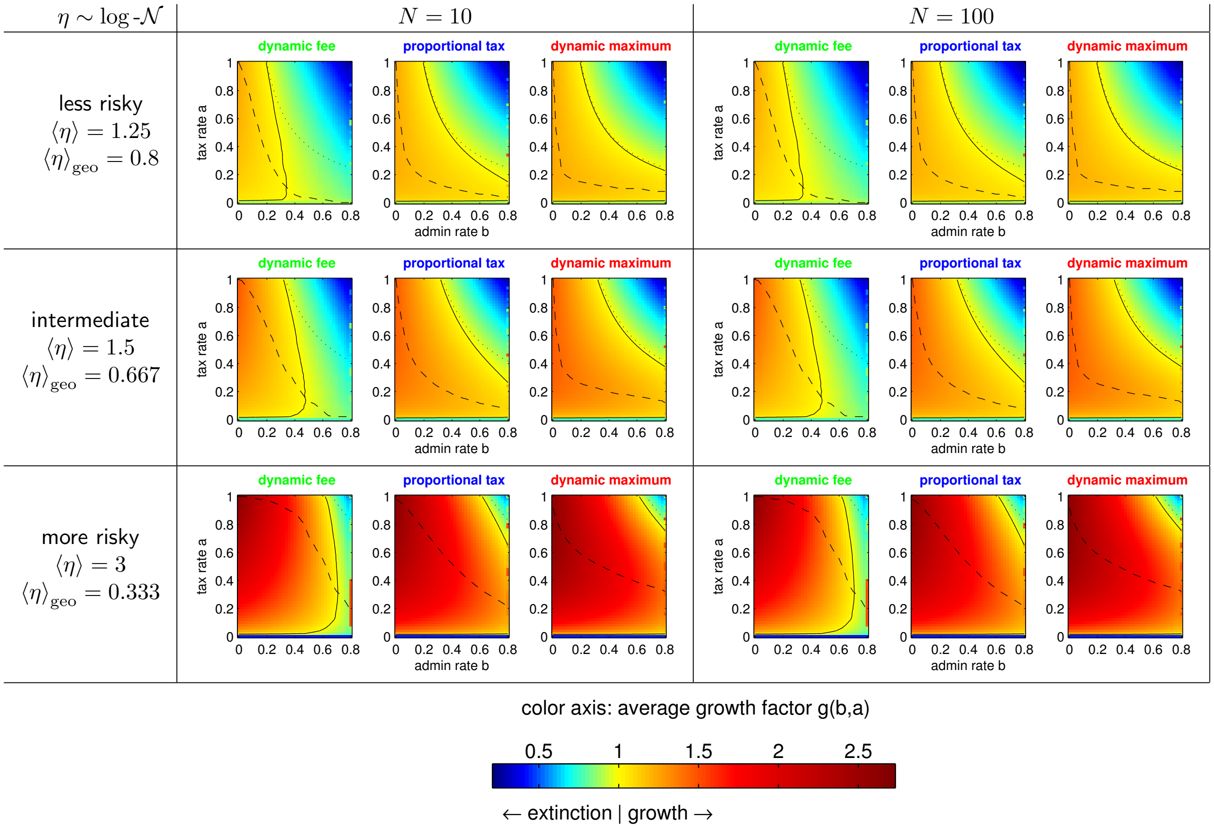

Figure 5. Simulation results analog to Figure 3**A** with less and more risky productions functions (in rows) and more agents (in another column).

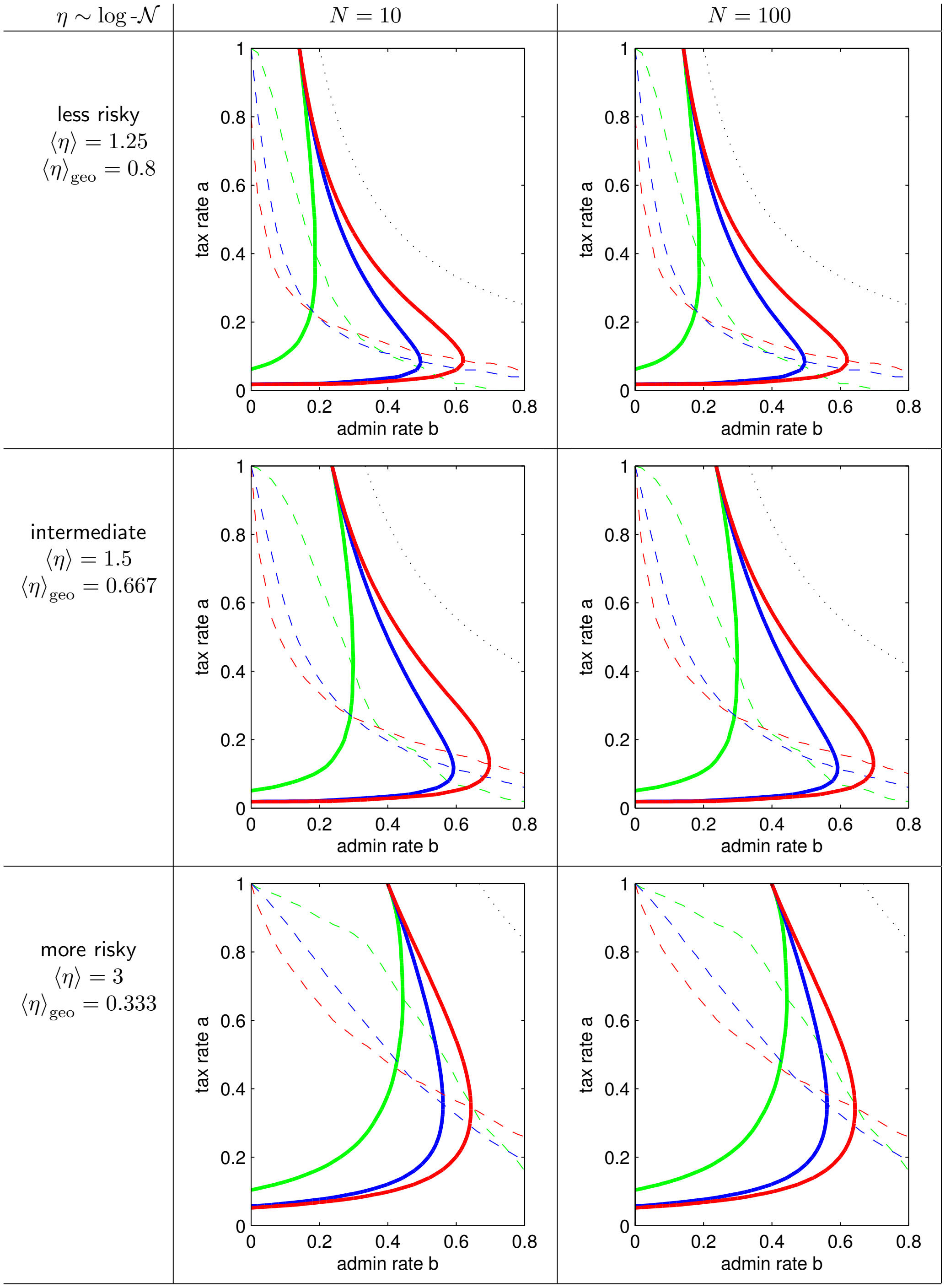

Figure 6. Simulation results analog to Figure 3**B** with less and more risky productions functions (in rows) and more agents (in another column). (Some lines as in Figure 7.)

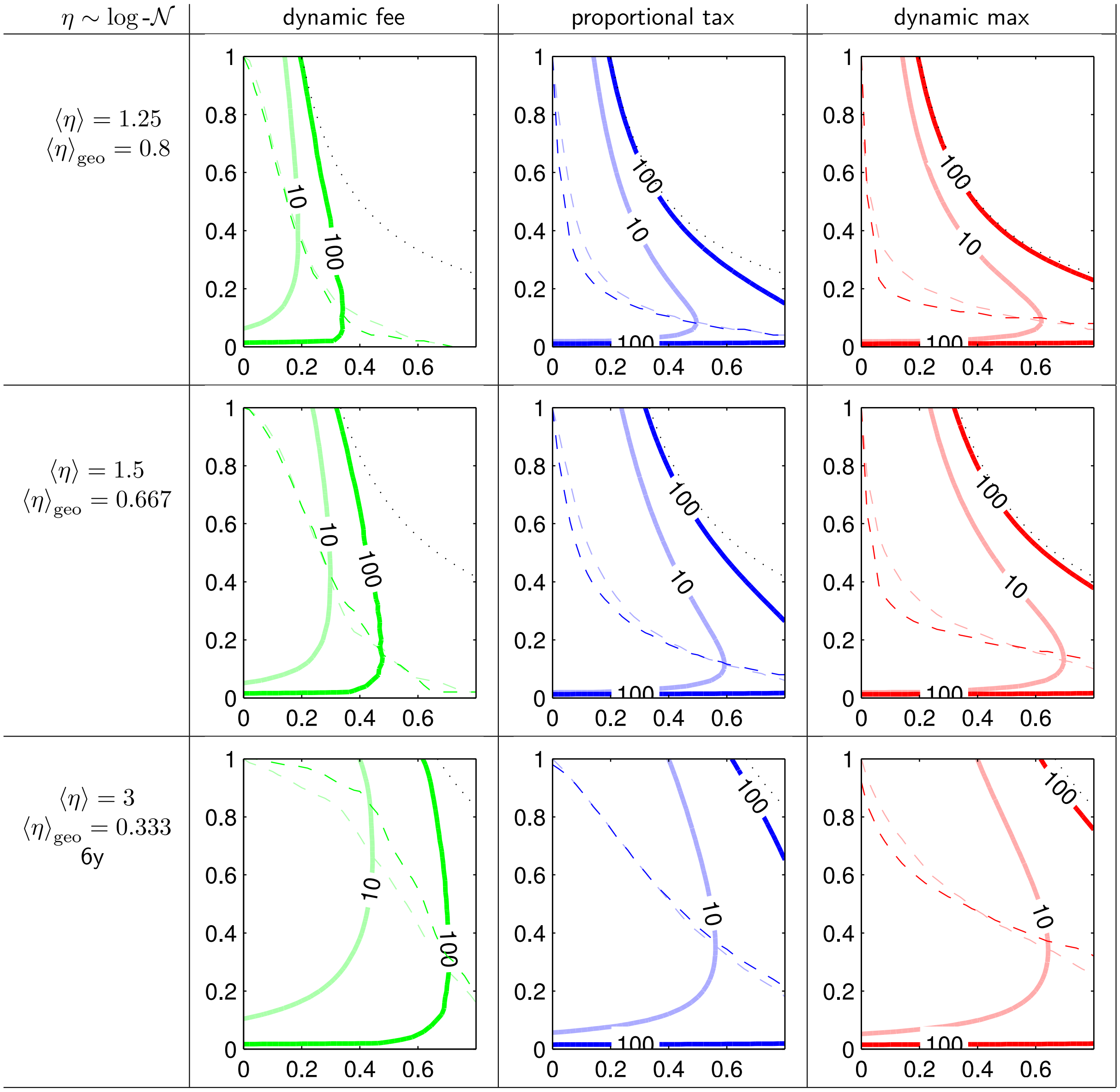

Figure 7. Same lines as in Figure 6, but such that different population sizes are in one plot and taxation schemes in columns.

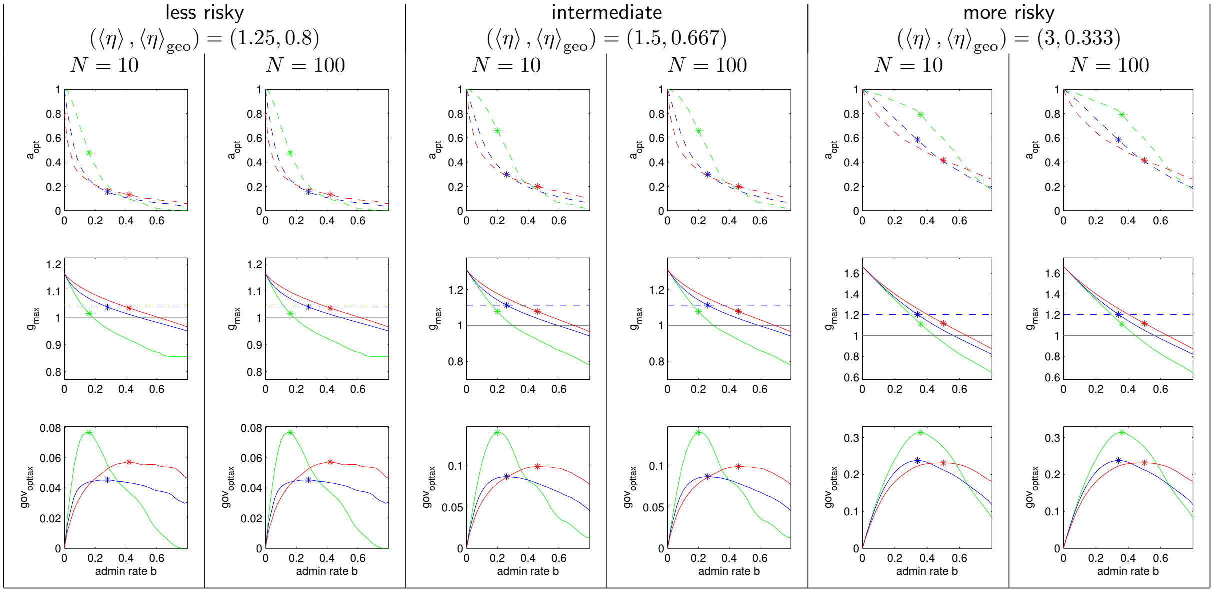

Figure 8. Simulation results analog to Figure 4 with less and more risky productions functions and more agents.

Table 1. Simulation setup.

independent variable	range	size of range
tax rate a	$0, \overset{+0.02}{\ldots}, 1$	51
admin rate b	$0, \overset{+0.02}{\ldots}, 0.8$	41
taxation scheme	dyn. fee, prop. tax, dyn. max	3
$(\langle\eta\rangle, \langle\eta\rangle_{\text{geo}})$	(1.25,0.8),(1.5,0.667),(3,0.333)	3
number of agents N	10, 100	2
$t_{\max}$	500	1

Numbers of parameter values multiply to 37,638 combinations. 100 simulation runs with $x_i(0) = 1$ were computed for each. Consequently the growth factor g was estimated regressing $\log(g)$ in $\log X(t) = \log N + t \cdot \log(g)$ (notice the intercept is naturally fixed at $\log N$). For these 3,763,800 values of g the geometric mean was computed for each combination over all 100 runs giving 37,638 *average growth factors* as the basis for Figures 3–8.