Convergence to Consensus by General Averaging

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Abstract. We investigate sufficient conditions for a discrete nonlinear non-homogeneous dynamical system to converge to consensus. We formulate a theorem which is based on the notion of averaging maps. Further on, we give examples that demonstrate that the theory of convergence to consensus is still not complete.

1 Introduction

We consider the problem of consensus formation under the action of general nonlinear averaging maps (or general means). We consider a set of agents $\underline{n} = \{1, \ldots, n\}$ where each of them has coordinates in a d-dimensional *opinion space* $S \subset \mathbb{R}^d$. The individual coordinates of agent i at time $t \in \mathbb{N}$ are labeled $x^i(t) \in S$, and $x(t) \in S^n \subset (\mathbb{R}^d)^n$ is called the *profile* at time $t \in \mathbb{N}$. Hence, we study discrete dynamical systems in $(\mathbb{R}^d)^n$ of the following form

$$x(t+1) = f_t(x(t)) \tag{1}$$

where $f_t : S^n \to S^n$. We denote the component functions by f_t^i. Accordingly we use upper indices for the number of the agents and lower indices for the dimension of the opinion space, e.g. x_k^i denotes the k-th component of the opinion of the i-th agent. We assume that the maps f_t are averaging maps (see below), and we are interested in conditions that ensure, that the solution converges to a consensus, i.e. there is γ such that $x^i(t) \to \gamma$ for every i. With matrices $A_t \in \mathbb{R}^{n \times n}$ we get a linear example

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by $f_t(x) = A_t x$. To recover averaging maps (as defined below) A_t is row-stochastic for all $t \in \mathbb{N}$. This problem is solved in special cases, e.g. when A is independent of t [9], in the case of finitely many A [10] or when the A_t's have positive diagonals, are type-symmetric and have positive minima uniformly bounded from below for all t [3, 6]. Here we treat the non-linear case similar to [5, 7, 8].

2 General Averaging Maps

In this section we define averaging maps (or general means, respectively). First we consider $d = 1$ and hence, an appropriate opinion space S is an interval. A *general mean* is a function $g : S^n \to S$ such that the *sandwich inequality* $\min_{i\in\underline{n}} x^i \le g(x) \le \max_{i\in\underline{n}} x^i$ holds. An example is the *power mean* for real $p \neq 0$ and positive x:

$$P_p(x) = (\frac{1}{n}((x^1)^p + \cdots + (x^n)^p))^{\frac{1}{p}}$$

The power mean includes the *arithmetic* ($p = 1$) and *harmonic* ($p = -1$) means. The *geometric mean* $\sqrt[n]{x^1 \ldots x^n}$ is approached for $p \to 0$, and $\max\{x^1,\ldots,x^n\}$ and $\min\{x^1,\ldots,x^n\}$ for $p \to \infty$ respectively $p \to -\infty$.

Further on, there are weighted means: for nonnegative numbers $\alpha_1,\ldots,\alpha_n$ which sum up to one there is the *weighted arithmetic mean* $\sum_{i=1}^n \alpha_i x^i$ and the *weighted geometric mean* $\prod_{i=1}^n x_i^{\alpha_i}$.

Another generalization is the *f-mean*. For a continuous and injective function $f : S \to \mathbb{R}$ (which is thus invertible on its range) the f-mean of $x^1,\ldots,x^n$ is

$$P_f(x) = f^{(-1)}(\frac{1}{n}\sum_{i=1}^n f(x^i)).$$

The power mean is represented here as $f(x) = x^p$, the geometric mean as $f(x) = \log(x)$. Of course, more means can be defined by means of means.

Now we extend the definition of a *general mean* to d-dimensional opinion vectors for $d \ge 2$. So let $S \subset \mathbb{R}^d$ be an appropriate opinion space. All means mentioned for the case $d = 1$ can be generalized to higher dimensions by taking them componentwise. Further on, one may define different one-dimensional means in each component.

What is a proper generalization of the one-dimensional sandwich inequality? We are going to answer this question with the help of *generalized barycentric coordinate maps*, but first we discuss two straightforward generalizations. First, a function $g : S^n \to S$ is called a *convex-hull mean* if it fulfills the *convex-hull sandwich inclusion*: $g(x) \in \mathrm{conv}_{i\in\underline{n}} x^i$. Obviously, the componentwise weighted arithmetic mean fulfills this property. But, e.g., the componentwise geometric mean does not (see Figure 1). Another possibility is, to use $\mathrm{cube}_{i\in\underline{n}} x^i := [\min_{i\in\underline{n}} x^i, \max_{i\in\underline{n}} x^i]$ instead of the convex hull. Notice that max and min are componentwise and that the interval is multidimensional. So cube represents the smallest closed hypercube in $\mathbb{R}^d$ which covers all vectors $x^1,\dots,x^n$. Figure 1 shows an example of the convex hull and the cube of a set of points in $\mathbb{R}^2$. A function $g : S^n \to S$ is called a *cube mean* if it fulfills the *cube sandwich inclusion*: $g(x) \in \text{cube}_{i\in\underline{n}} x^i$.

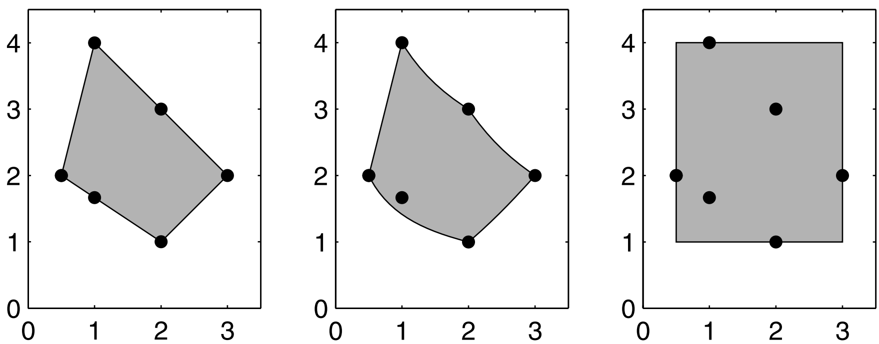

Fig. 1 A set of six points in $\mathbb{R}^2_{\geq 0}$. The gray area is their convex hull (left), all possible weighted geometric means (center) and their cube (right).

The convex-hull means and the cube means can be generalized further with the help of a generalized *barycentric coordinate map*. To motivate the construction note that the cube is indeed the convex hull of the 2^d points which lie at its vertices. Hence, we could write $\text{cube}_{i\in\underline{n}} x^i = \text{conv}_{i\in\underline{2^d}} y^i(x)$ with an appropriate map $y : \mathbb{R}^n \to \mathbb{R}^{2^d}$ which takes n points in $\mathbb{R}^d$ and maps them to the 2^d possible combinations of taking componentwise max and min. To formalize this further we call $y : S^n \to S^m$ a generalized *barycentric coordinate map* if for every $k \in \underline{n}$ it holds that $x^k \in \text{conv}_{i\in\underline{m}} y^i(x)$. It is now natural to call $\text{conv}_{i\in\underline{m}} y^i(x)$ the *y-convex hull of x*. So, a y-convex hull is a set-valued function from S^n to the compact and convex subsets of S. We call a set *y-convex*, if it is the union of the y-convex hulls of all n of its points. A function g is a *y-convex hull mean* if $g(x) \in \text{conv}_{i\in\underline{m}} y^i(x)$. Note that the convex hull is obtained from the y-convex hull with $m = n$ and y the identity. The cube is obtained with $m = 2^d$ and an appropriate mapping y. Many other examples fit into this setting: the smallest interval for any basis of $\mathbb{R}^d$ [1, Example 2], or the smallest polytope with faces parallel to a set of $k \geq d+1$ hyperplanes [1, Example 3] containing $x^1,\dots,x^n$ (the generalized barycentric coordinates are then the extreme points of the polytope, perhaps with multiplicities e.g. if the number of extreme points is smaller than n). Now, we define the central notion of this paper.

Definition 1. Let $S \subset \mathbb{R}^d$, $y : S^n \to S^m$ be a generalized barycentric coordinate map such that S is y-convex. A mapping $f : S^n \to S^n$ is called a *y-averaging map*, if for every $x \in S^n$ it holds

$$\text{conv}_{i\in\underline{m}} y^i(f(x)) \subset \text{conv}_{i\in\underline{m}} y^i(x). \tag{2}$$

Furthermore, a *proper y-averaging map* is a y-averaging map, such that for every $x \in S^n$ which is not a consensus, the above inclusion is strict.

3 Convergence to Consensus

To state a sufficient condition on the maps f_t which ensure that the solution of (1) converge to consensus we need a little more notation. Let $d(x,C)$ denote the distance of a point x and a compact set C. The *Hausdorff metric* for compact non-empty sets is defined as

$$d_H(B,C) := \max\{\sup_{b\in B} d(b,C), \sup_{c\in C} d(c,B)\}.$$

If $B \subset C$ it holds $d_H(B,C) := \sup_{b\in B} d(b,C)$.

The next notion we need, is 'equiproper averaging map':

Definition 2. Let y be a generalized barycentric coordinate map and let F be a family of proper y-averaging maps. F is called *equiproper*, if for every $x \in S^n$ which is not a consensus, there is $\delta(x) > 0$ such that for all $f \in F$

$$d_H\big(\mathrm{conv}_{i\in\underline{m}} y^i(f(x)), \mathrm{conv}_{i\in\underline{m}} y^i(x)\big) > \delta(x). \tag{3}$$

We can state the following lemma:

Lemma 1. *Let f_t be a sequence of averaging maps forming an equiproper family of averaging maps such that $f_t \to g$ pointwise. Then g is a proper averaging map.*

Proof. First we show that g is an averaging map. Take $x \in S^n$ and let $\varepsilon > 0$, due to convergence of $(f_t)_i$ to g_i and continuity of y there is t_0 such that for all $t > t_0$ it holds $\|y^i(f_t(x)) - y^i(g(x))\| < \varepsilon$. Due to $y^i(f_t(x)) \in \mathrm{conv}_{i\in\underline{m}} y^i(x)$ it follows that the maximal distance of $y^i(g(x))$ to $\mathrm{conv}_{i\in\underline{m}} y^i(x)$ is less than ε, and thus $y^i(g(x)) \in \mathrm{conv}_{i\in\underline{m}} y^i(x)$ because $\mathrm{conv}_{i\in\underline{m}} y^i(x)$ is closed.

Now we show that g is proper. To this end, let $x \in S^n$ be not a consensus. We have to show that there is $z^* \in \mathrm{conv}_{i\in\underline{m}} y^i(x)$ but $z^* \notin \mathrm{conv}_{i\in\underline{m}} y^i(g(x))$. (Note that $z^* \in S$, while $x \in S^n$ and $y(x) \in S^m$.) We know that there is for each $t \in \mathbb{N}$ an $z(t) \in \mathrm{conv}_{i\in\underline{m}} y^i(x)$ with $z(t) \notin \mathrm{conv}_{i\in\underline{m}} y^i(f_t(x))$. According to the equiproper property it can be chosen such that the distance of $z(t)$ to $\mathrm{conv}_{i\in\underline{m}} y^i(f_t(x))$ is bigger than $\delta(x)/2 > 0$ for all $t \in \mathbb{N}$. Further on, we know that the set difference $\mathrm{conv}_{i\in\underline{m}} y^i(f_t(x)) \backslash \mathrm{conv}_{i\in\underline{m}} y^i(x)$ is non empty and bounded, thus there is a subsequence t_s such that $z(t_s)$ converges to an $z^* \in \mathrm{conv}_{i\in\underline{m}} y^i(x)$. Because of the construction it also holds $z^* \notin \mathrm{conv}_{i\in\underline{m}} y^i(g(x))$. □

Lemma 2. *Let $x(t)$ be a solution of (1) and let $C(t) = \mathrm{conv}_{i\in\underline{m}} y^i(x(t))$. There exists a subsequence t_s such that $x(t_s) \to c$ for $s \to \infty$ and it holds $c^i \in C = \cap_t C(t)$ for every i.*

Proof. Due to compactness of $C(0)^n$ there exists a convergent subsequence of $x(t)$ and we call its limit c. It remains to show that $c^i \in C$. Let $T \in \mathbb{N}$ and $\varepsilon > 0$.

Take s large enough, to have $\|x(t_s) - c\| \leq \varepsilon$. If we take s even larger, we have $x^i(t_s) \in C(T)$ which shows $c^i \in C(T)$ since $C(T)$ is closed. Hence, $c^i \in C(T)$ for every T. □

Before we state our main theorem we need one more notion: We call a family F of continuous functions *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $f \in F$ it holds that $\|x - x'\| \leq \delta$ implies $\|f(x) - f(x')\| \leq \varepsilon$. Note that δ is chosen independently of f. Now, the main theorem is as follows:

Theorem 1. *Let $S \subset \mathbb{R}^d$, y be a generalized barycentric coordinate map such that S is y-convex, and F be an equicontinuous family of equiproper y-averaging maps on S^n. Then it holds for any sequence $(f_t)_{t\in\mathbb{N}}$ with $f_t \in F$ and any $x(0) \in S^n$ that the solution of (1) converges to a consensus.*

Proof. In the first step one shows that C from Lemma 2 fulfills $C = \operatorname{conv}_{i\in\underline{m}} y^i(c)$. This step uses that, due to the Theorem of Arzelà-Ascoli and Lemma 1, there is a uniformly convergent subsequence of f_t with a proper y-averaging map g as a limit. Now one shows that $\operatorname{conv}_{i\in\underline{m}} y^i(g(c)) = \operatorname{conv}_{i\in\underline{m}} y^i(c)$ which implies that c is a consensus since g is proper. For details we refer to [5].

It remains to show that the whole sequence $x(t)$ converges to $c = (\gamma, \ldots, \gamma)$. This can be seen as follows: For $\varepsilon > 0$ there exists s_0 such that for $s > s_0$ we have $\|y^i(x(t_s)) - \gamma\| \leq \varepsilon$. Moreover, for $t > t_{s_0}$ we have $x(t) \in C(t_{s_0})$ and hence $x^i(t) = \sum_j a^j y^j(x(t_{s_0}))$ is convex combination. We conclude

$$\|x^i(t) - \gamma\| = \|\sum_j a^j (y^j(x(t_{s_0})) - \gamma)\| \leq \sum_j \|y^j(x(t_{s_0})) - \gamma\| \leq m\varepsilon \tag{4}$$

and hence, $x(t) \to c = (\gamma, \ldots, \gamma)$. □

The proof follows the lines of the main theorem in [4] which now appears as a corollary since there the system $x(t+1) = f(x(t))$ with just one proper averaging map f is considered. Some more corollaries can be deduced.

Corollary 1. *Let $F = \{f_1, \ldots, f_m\}$ be a finite family of proper y-averaging maps on $S^n \subset (\mathbb{R}^d)^n$, with S an appropriate opinion space. Let F be uniformly continuous. Then it holds for a sequence $(f_t)_{t\in\mathbb{N}}$ with $f_t \in F$ and $x(0) \in S^n$ that the solution of (1) converges to consensus.*

Proof. Since the family is finite, it is uniformly continuous and equiproper. □

Corollary 2. *Let F be a family of averaging maps on $S^n \subset (\mathbb{R}^d)^n$, with S an appropriate opinion space. Let F be uniformly equicontinuous and at least one element in F is proper and all proper elements of F are equiproper. Let $(f_t)_{t\in\mathbb{N}}$ be a sequence with $f_t \in F$ and t_s be a subsequence such that f_{t_s} is proper. Then, for $x(0) \in S^n$, the solution of (1) converges to consensus.*

Proof. Theorem 1 for the sequence f_{t_s} gives subsequential convergence to consensus. An estimate similar to (4) gives convergence of the whole sequence. □

The proof of the next corollary uses similar techniques.

Corollary 3. *Let F be a family of averaging maps on $S^n \subset (\mathbb{R}^d)^n$, with S an appropriate opinion space. Let F be uniformly equicontinuous. Let $(f_t)_{t\in\mathbb{N}}$ be a sequence with $f_t \in F$ and $(t_s)_{s\in\mathbb{N}}$ be a subsequence (with $t_0 = 0$) such that the $f_s := f_{t_{s+1}-1} \circ f_{t_{s+1}-2} \circ \cdots \circ f_{t_s+1} \circ f_{t_s}$ are equiproper. Then, for $x(0) \in S^n$, the solution of (1) converges to consensus.*

In the spirit of [1] we state another generalization of Theorem 1. The generalization deals with deformations of the hull. To this end, let $S,T \subset \mathbb{R}^d$ be compact and $\phi : T \to S$ be a homeomorphism. For a generalized barycentric coordinate map $y : S^n \to S^m$ we define the y,ϕ-hull as $\phi^{-1}(\mathrm{conv}_{i\in\underline{m}} y^i(\phi(x)))$. Now, a y,ϕ-averaging map g is defined analogous to Definition 1:

$$\phi^{-1}(\mathrm{conv}_{i\in\underline{m}} y^i(\phi(g(x)))) \subset \phi^{-1}(\mathrm{conv}_{i\in\underline{m}} y^i(\phi(x))).$$

Note, that the y,ϕ-hull is not necessarily convex, see [1, Example 6]. The extension of the notions 'proper' and 'equiproper' is straightforward. For the proof of the following theorem we refer to [5].

Theorem 2. *Let $\phi : T \to S$ be continuous with Lipschitz continuous inverse and let y be a generalized barycentric coordinate map such that S is y-convex. Let G be a family of equicontinuous, equiproper y,ϕ-averaging maps on T^n. Then it holds for any sequence $(g_t)_{t\in\mathbb{N}}$ with $g_t \in G$ and any $x(0) \in T^n$ that the solution of $x(t+1) = g_t(x(t))$ converges to a consensus.*

4 Examples and Counterexamples

We give some examples that illustrate the role of the different assumptions in Theorem 1.

Example 1. Let $f : (\mathbb{R}_{\geq 0})^2 \to (\mathbb{R}_{\geq 0})^2$ with

$$f(x^1,x^2) := \begin{cases} \left(\frac{3}{4}x^1 + \frac{1}{4}x^2, \frac{3}{4}x^1 + \frac{1}{4}x^2\right) & \text{if } x^1 + x^2 > 10, \\ \left((x^1)^{\frac{3}{4}}(x^2)^{\frac{1}{4}}, (x^1)^{\frac{1}{4}}(x^2)^{\frac{3}{4}}\right) & \text{otherwise.} \end{cases}$$

This averaging map converges to consensus but is not continuous. For example for $x(0) = (1,9)$ it converges to $(3,3)$ but for $x(0) = (1+\varepsilon,9)$ it converges to $(5+\varepsilon/2, 5+\varepsilon/2)$. So continuity is not necessary for convergence.

Example 2. Let $f : (\mathbb{R})^3 \to (\mathbb{R})^3$ with

$$f(x^1,x^2,x^3) := \begin{cases} (x^1,x^2,\frac{1}{2}x^3 + \frac{1}{2}\min\{x^1,x^2\}) & \text{if } x^3 < \min\{x^1,x^2\}, \\ (x^1,x^1,x^1) & \text{otherwise.} \end{cases}$$

Starting with $x(0) = (2,3,1)$ the discrete dynamical system $x(t+1) = f(x(t))$ will converge to $(2,3,2)$, although it is a proper averaging map. But it is not continuous at all points where $x^3 = \min\{x^1, x^2\}$ and $x^1 \neq x^2$.

Example 3. Let $f_t : (\mathbb{R})^2 \to (\mathbb{R})^2$ with

$$f_t(x^1,x^2) := \left((1-4^{-t})x^1 + 4^{-t}x^2,\ 4^{-t}x^1 + (1-4^{-t})x^2\right).$$

It is easy to see that theses f_t's are proper and that for $t \geq 1$ and $x(1) = (0,1)$ it holds that $x^1(t) < \frac{1}{3}$ and $x^2(t) > \frac{2}{3}$. Obviously, $\{f_t \,|\, t \in \mathbb{N}\}$ is not equiproper because f_t converges to the identity as $t \to \infty$.

Example 4. Let $f_t : (\mathbb{R})^2 \to (\mathbb{R})^2$ with

$$f_t(x^1,x^2) := \left((1-\tfrac{1}{t})x^1 + \tfrac{1}{t}x^2\ ,\ x^2\right).$$

This example is not equiproper, because f_t converges to the identity for $t \to \infty$. But for $t \geq 2$ and any initial values $(x^1(2),x^2(2)) \in (\mathbb{R})^2$ the system $x(t+1) = f_t(x(t))$ has the solution $x(t) = (\frac{1}{t-1}x^1(2) + \frac{t-2}{t-1}x^2(2)\ ,\ x^2(2))$ and thus converges to consensus at $x^2(2)$ for all initial values.

Of course, equicontinuity is not necessary. Example 1 gives a one-element family of not equicontinuous averaging maps which converge. Now, we show that a family of uniformly continuous proper averaging maps is not enough to ensure convergence. The example is inspired by bounded confidence.

Example 5 (Vanishing confidence). Let $f_t : \mathbb{R}^n \to \mathbb{R}^n$ with

$$(f_t)_i(x) := \frac{\sum_{j=1}^n D_t(|x^i - x^j|)x^j}{\sum_{j=1}^n D_t(|x^i - x^j|)}$$

and $D_t : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Now, f_t is an averaging map for any choice of D_t. Further on, f_t is continuous if D_t is, and f_t is proper if D_t is strictly positive. The Hegselmann-Krause model [2, 7] with homogeneous bound of confidence $\varepsilon > 0$ comes out for D_t being a non-continuous *cutoff function*

$$D_t(y) = \begin{cases} 1 \text{ if } y \leq \varepsilon \\ 0 \text{ otherwise.} \end{cases}$$

We chose $D_t(y) := e^{-(\frac{y}{\varepsilon})^t}$ as a sequence of functions which has the cutoff function as a limit function. So, D_t is continuous but $\{D_t \,|\, t \in \mathbb{N}\}$ is not equicontinuous. Rough estimates show that with $x(0) = (0,8)$, $\varepsilon = 1$ the process $x(t) = f_t(x(t))$ does not converge to consensus although only proper averaging maps are involved.

5 Comparison with a Theorem of Moreau

Theorem 1 is similar to a theorem of Moreau [8, Theorem 2]. We cite it here to discuss similarities and differences. It incorporates changing communication networks into self-maps with averaging properties.

Theorem 3 (Moreau). *Let $(f_t)_{t\in\mathbb{N}}$ be a sequence of self maps on $S^n \subset (\mathbb{R}^d)^n$, with S convex and closed. Let $(N(t))_{t\in\mathbb{N}}$ be a communication regime where all networks have positive diagonals: Moreover, assume thatthere is $T \in \mathbb{N}$ such that for all $t_0 \in \mathbb{N}$ the network $\operatorname{inc}(\sum_{t=t_0}^{t_0+T} N(t))$ has only one essential class*[1].

Further on, it should exist for each network with positive diagonal N, each $x \in S^n$ and each agent $k \in \underline{n}$ a compact set $e_k(x,N)$ such that

1. *For all $t \in \mathbb{N}$ it holds $(f_t)_k(x) \in e_k(x,N)$,*
2. *$e_k(x,N) \subset \operatorname{ri}\operatorname{conv}_{i\in \operatorname{nb}(k,N)}\{x^i\}$,*
3. *$e_k(x,N)$ depends continuously on x.*

Then it holds for $x(0) \in S^n$ that the solution of (1) converges to a consensus.

The theorem has been significantly reformulated in comparison with the original to fit it in our vocabulary. Especially the original theorem is about "uniform global attractivity of the system with respect to the set of equilibrium solutions $x^1 = \cdots = x^n =$ constant" which is equivalent to convergence to consensus for every $x(0) \in S^n$.

Items 1 and 2 in the assumptions of Theorem 3 are similar to a 'proper convex hull averaging map with respect to the current network'. It is averaging due to conv, and proper due to ri (actually ri is a stronger assumption than proper). The continuity assumption in item 3 shows similarity to the assumption of equicontinuity in Theorem 1. Equiproper from Theorem 1 finds its analog in Theorem 3 in the fact that in item 1 it holds $(f_t)_k(x) \in e_k(x,N)$ and e_k is independent of t. Especially, the assumption that the e_k's are in the relative interior of convex hulls is more strict than the assumption of properness. But, the theorems can not be compared directly. Moreau's Theorem allows changing communication topologies and poses assumptions on this. Our theorem does not deal with communication networks, but with arbitray switching update maps from an equiproper set.

Example 6. Let $g_1, g_2, g_3, g_4 : (\mathbb{R}^d)^3 \to \mathbb{R}^d$ with $g_1(x) := \max\{x^1, x^2, x^3\}$, $g_2(x) := \frac{1}{3}(x^1 + x^2 + x^3)$, $g_3(x) := \sqrt[3]{x^1x^2x^3}$ and $g_4(x) := \min\{x^1, x^2, x^3\}$ be general multidimensional means (all computations componentwise) and $f^{\sigma_1\sigma_2\sigma_3} : (\mathbb{R}^d)^3 \to (\mathbb{R}^d)^3$ with $f^{\sigma_1\sigma_2\sigma_3} := (g_{\sigma_1}, g_{\sigma_2}, g_{\sigma_3})$ be averaging maps. Now it is easy to verify, that

$$F := \{f^{\sigma_1\sigma_2\sigma_3} \mid (\sigma_1,\sigma_2,\sigma_3) \in \{1,2,3,4\}^3 \text{ but 1 and 4 not both in } (\sigma_1,\sigma_2,\sigma_3)\}$$

is an equicontinuous and equiproper set of averaging maps w.r.t cube. Thus, for any sequence f_t with elements from F and $x(0) \in (\mathbb{R}^d)^3$ it holds that $x(t+1) = f_t(x(t))$

[1] With inc we denote the incidence matrix, i.e. $\operatorname{inc}(A)_{i,j} = 1$ if $A_{i,j} \neq 0$ and 0 otherwise.

converges to consensus due to Theorem 1. Theorem 3 is not applicable because item 2 does not hold for all elements of F.

References

1. Angeli, D., Bliman, P.A.: Stability of leaderless multi-agent systems. Extension of a result by Moreau. Mathematics of Control, Signals & Systems 18(4), 293–322 (2006)
2. Hegselmann, R., Ulrich Krause, U.: Opinion Dynamics Driven by Various Ways of Averaging. Computational Economics 25(4), 381–405 (2004)
3. Hendrickx, J.M., Blondel, V.D.: Convergence of Different Linear and Non-Linear Vicsek Models. CESAME research report 2005.57 (2005)
4. Krause, U.: Compromise, consensus, and the iteration of means. Elemente der Mathematik 63, 1–8 (2008)
5. Lorenz, D.A., Lorenz, J.: On conditions for convergence to consensus. arXiv.org/abs/0803.2211 (March 2008)
6. Lorenz, J.: A Stabilization Theorem for Dynamics of Continuous Opinions. Physica A 355(1), 217–223 (2005)
7. Lorenz, J.: Repeated Averaging and Bounded Confidence – Modeling, Analysis and Simulation of Continuous Opinion Dynamics. PhD thesis, Universität Bremen (March 2007)
8. Moreau, L.: Stability of Multiagent Systems with Time-Dependent Communication Links. IEEE Transactions on Automatic Control 50(2) (2005)
9. Seneta, E.: Non-Negative Matrices and Markov Chains, 2nd edn. Springer, Heidelberg (1981)
10. Wolfowitz, J.: Products of Indecomposable, Aperiodic, Stochastic Matrices. In: Proceedings of the American Mathematical Society Eugene, vol. 15, pp. 733–737 (1963)