Advances in Complex Systems, Vol. 10, No. 2 (2007) 251–269

ABOUT THE POWER TO ENFORCE AND PREVENT CONSENSUS BY MANIPULATING COMMUNICATION RULES

JAN LORENZ

Department of Mathematics and Computer Science,
Universität Bremen, Bibliothekstraße,
28359 Bremen, Germany
math@janlo.de

DIEMO URBIG

School of Business and Economics and Department of Computer Science,
Humboldt-Universität zu Berlin, Unter den Linden 6,
10099 Berlin, Germany
diemo@urbig.org

Received 7 January 2006
Revised 24 May 2006

We explore the possibilities of enforcing and preventing consensus in continuous opinion dynamics that result from modifications in the communication rules. We refer to the model of Weisbuch and Deffuant, where n agents adjust their continuous opinions as a result of random pairwise encounters whenever their opinions differ not more than a given bound of confidence ε. A high ε leads to consensus, while a lower ε leads to a fragmentation into several opinion clusters. We drop the random encounter assumption and ask: How small may ε be such that consensus is still possible with a certain communication plan for the entire group? Mathematical analysis shows that ε may be significantly smaller than in the random pairwise case. On the other hand, we ask: How large may ε be such that preventing consensus is still possible? In answering this question, we prove Fortunato's simulation result that consensus cannot be prevented for $\varepsilon > 0.5$ for large groups. Next, we consider opinion dynamics under different individual strategies and examine their power to increase the chances of consensus. One result is that balancing agents increase chances of consensus, especially if the agents are cautious in adapting their opinions. However, curious agents increase chances of consensus only if those agents are not cautious in adapting their opinions.

Keywords: Continuous opinion dynamics; bounded confidence; communication structure; balancing agents; curious agents.

1. Introduction

What happens if people meet and discuss their opinions regarding a political party, a brand, or a new product? Generally, when people meet they influence one another and as a consequence may change their opinions. Such opinion formation processes are at the heart of models that explain voting behavior as well as models of innovation diffusion (see, for instance, Ref. 1).

If people who are assumed to have opinions toward something meet and discuss, they may adapt their opinions towards the other agent's opinion and reach a compromise, they may move away from consensus when their initial positions are too different, or they could ignore each other. For simplification, we only consider one-dimensional opinions such that they can be represented by real numbers between zero and one. We will only examine compromising agents under bounded confidence, which implies that individuals who differ too much in their opinions do not affect, and thus ignore, each other. This assumption mirrors the psychological concept of selective exposure, where people tend to perceive their environment in favor of their own opinions and thereby avoid communication with people with conflicting opinions. However, if agents do not ignore each other, then they get closer in their opinions. Such systems of agents, who update their opinions via averaging with other sufficiently similar opinions, are referred to as systems of *continuous opinion dynamics under bounded confidence*. Models following this paradigm have been proposed by Hegselmann and Krause [2,3] and Weisbuch, Deffuant, and others [4,5]. In the Hegselmann and Krause model (HK model) every agent perceives the opinions of every other agent and builds his new opinion as an average of sufficiently close opinions. Thereby, Hegselmann and Krause added the assumption of bounded confidence to a previous linear opinion dynamics model by DeGroot [6,7]. Hegselmann and Krause's main question was what conditions related to the bounded confidence, in other words the degree of open mindedness, were necessary for a consensus to be reached.

While for Hegselman and Krause all agents interact simultaneously, the agents in the model by Weisbuch and Deffuant (WD model) engage in random pairwise encounters. Several other extensions (e.g. in Ref. 8) and a combination of both models, the HK model and the WD model [9], have been analyzed. A model which includes the centrifugal forces of rejecting agents has been proposed by Jager [10]. Opinion dynamics models have also been examined in incompletely linked networks, for instance, in scale-free networks [11,12].

While the conditions necessary for consensus have already been examined regarding the bounded confidence, we will explore conditions affecting the rules of communication in the sense of who talks with whom. Even if we regard a completely linked society as given and thus look at the WD model, this model has an unexplored free parameter in the order of who communicates with whom at what time. We will call rules that modify this order the *communication regime*. Studying this parameter is the aim of this paper. Considering the complexity of human organizations and the different institutions that foster or manipulate the communication regime makes immediately clear why this question is of relevance. We will see that, although the bounds of confidence have a significant impact, also the factors that control the communication regime significantly affect the emergence of consensus or dissent. Thus, our two leading questions are: To what extent does the communication regime matter? Do individual communication rules, like being balancing or being curious, matter? To focus our analysis, we concentrate on possibilities and strategies to foster or prevent consensus.

The above-mentioned models on opinion dynamics, i.e. the WD model and the HK model, were previously studied with the more general technique of differential equations on density based state spaces instead of single agents in finite populations [13–15]. However, we will apply these questions to populations of finite size, more precisely, less than one thousand members. This prevents us from using such general techniques that abstract from single agents. Nevertheless, it is interesting because it is a more realistic assumption. The model assumes a completely mixed population where everybody has the same chance of interacting with everybody else. However, in human societies the size of groups, where one can reasonably assume a complete mixing, does not scale arbitrarily. For instance, Zhou *et al.* [16] and Hill and Dunbar [17] argue that some group sizes are more frequently observed than others and that at a certain critical number, groups exhibit significantly different properties in, for instance, their communication patterns. The organizational literature also suggests that beyond critical sizes, hierarchies will be established. Furthermore, the distribution of people across different geographical locations also restricts the set of potential interaction partners. All these aspects suggest that for very large systems the assumption of completely mixed societies is strongly violated. We believe that these arguments demonstrate the necessity of investigating finitely sized groups if one sticks to the complete mixing assumption.[a] Although finite size is often associated with a difficult analysis, we will demonstrate that for opinion dynamics this approach in some circumstances still lends itself to an analytical approach.

After a short introduction of the Weisbuch and Deffuant model, we answer the question concerning the extent to which the communication regime is able to enforce or prevent consensus. We will see that the results of Deffuant, Weisbuch, and others are not robust against manipulation of the communication order. The result on preventing consensus supports Fortunato's claim of universality of the threshold for complete consensus [11]. Fortunato provides simulation-based evidence that consensus is reached for $\varepsilon > 0.5$ irrespective of the structure of an underlying connected social network. Our result will explicitly define a threshold such that for larger bounds of confidence consensus cannot be prevented. In this way, the simulation results by Fortunato are formally proved without any simulation, but in the limit of large numbers of agents and uniformly distributed initial opinions. However, we also show how the result differs for populations of different finite sizes.

[a]One should be aware that research regarding certain social and economic phenomena are only based on the assumption of finite sizes, e.g. theory on competition among firms. In fact, the infinite size assumption may sometimes represent the most uninteresting case. As such, we suggests that in the social sciences the infinite size assumption should not be treated as the undiscussed default. At the very least, a justification that violating the finite size assumption does not cause a major change in system behavior is critical.

For instance, for ε below a specific level there is a zero probability of consensus in finite populations, while the level depends on the groups size and the cautiousness of agents.

The communication regime we construct in Sec. 3 to reach the extreme bounds for preventing and enforcing consensus relies on full knowledge of the opinions of all agents. To circumvent this, in Sec. 4 we run a simulation analysis with individual strategies, where agents are either balancing or curious. This only requires that agents know their individual recent communication history. "Balancing" means that an agent who has talked with somebody who has a higher opinion seeks later on somebody with a lower one. "Curious" means that agents seek partners with opinions in the same direction as those of their former communication partners. Particularly, the interplay of these strategies with the cautiousness that agents exhibit is interesting. We will see that these very simple communication strategies, which could reasonably be applied by humans, can significantly increase the chances for consensus.

2. Dynamics of Continuous Opinions

We analyze the model of continuous opinion dynamics that was introduced by Weisbuch, Deffuant, and others [4,5]. The dynamics are driven by random encounters of two agents, who compromise if their distance in opinions is below a certain bound of confidence ε. The model always converges to a stabilized opinion formation, where agents in the same cluster have the same opinion in the long run [18].

We consider $n \in \mathbb{N}$ agents, who each have an opinion that is represented by a real number. The opinion of agent $i \in \underline{n} := \{1, \ldots, n\}$ at timestep $t \in \mathbb{N}_0$ is represented by $x_i(t) \in \mathbb{R}$. We call the vector $x(t) \in \mathbb{R}^n$ the *opinion profile* at timestep t.

Definition 1 (WD Model). Given an initial opinion profile $x(0) \in \mathbb{R}^n$, a bound of confidence $\varepsilon \in \mathbb{R}_{>0}$, and a cautiousness parameter[b] $\mu \in]0, 0.5]$ we define the *WD model* as a process of opinion dynamics as the random process $(x(t))_{t\in\mathbb{N}_0}$ that chooses in each timestep $t \in \mathbb{N}_0$ two agents i and j randomly and equally distributed from the set of agents $\underline{n}$. Agents i and j perform the action

$$\begin{aligned}
&\text{if } |x_i(t) - x_j(t)| < \varepsilon \\
&\qquad x_i(t+1) = (1-\mu)x_i(t) + \mu x_j(t), \\
&\qquad x_j(t+1) = \mu x_i(t) + (1-\mu)x_j(t), \\
&\text{else} \\
&\qquad x_i(t+1) = x_i(t), \quad x_j(t+1) = x_j(t).
\end{aligned}$$

The bound of confidence ε was previously shown to be the most significant parameter to control the number of emerging clusters. For randomly distributed initial profiles with opinions between zero and one $x \in [0,1]^n$ and $n = 1000$ it is

[b]This parameter is called convergence parameter in Ref. 4.

shown via simulations that consensus is reached in nearly every case for $\varepsilon > 0.3$ [4]. For lower ε, the usual outcome is polarization into a certain number of opinion clusters. Weisbuch, Deffuant *et al.* derived by computer simulation the "$1/2\varepsilon$-rule," which states that the number of surviving clusters is roughly the integer part of $1/2\varepsilon$.[c]

The cautiousness parameter μ had been considered to have no effect on clustering in the basic model (only on convergence time) [4, 5]. However, there is already some evidence that μ can affect the clustering as well as that the effect of μ interacts with other parameters, e.g. number of agents that participate in an interaction [9, 19]. Furthermore, different random initial profiles may lead to different numbers of clusters, and even the same initial profile may lead to different numbers of clusters for different random choices of communicating pairs. In most previous studies, the dependence on the initial profile and on the communication regime is not considered due to the randomness assumption. In the next section, we will incorporate both initial opinion profile and communication regime to examine the bounds for enforcing and preventing consensus.

3. Enforcing and Preventing Consensus

In this section, we give mathematical answers to the questions: How small may ε be such that enforcing consensus is still possible? How large may ε be such that preventing consensus is still possible?

Let our initial opinion profile $x(0)$ and the parameter μ be fixed. We define ε_{low} as the smallest value of epsilon for which there is a communication regime that leads to a consensus. Obviously, ε_{low} depends on the initial opinion profile and perhaps on μ. We will give a lower and an upper limit for ε_{low} based on a communication regime that looks like a phone chain of those people with the most similar opinions, or in other words a *phone chain of closest*.

For our approximation, we must take a detailed look at the initial opinion profile. For this reason, we regard our initial opinion profile $x(0)$ as ordered such that $x_1(0) \leq \cdots \leq x_n(0)$, without loss of generality. For our considerations, it is useful to look at the gaps between the opinions. We define for $i \in \underline{n-1}$ the gap to the next neighbor as $\Delta x_i(t) := x_{i+1}(t) - x_i(t)$. If we regard an opinion profile as a function $x_{(\cdot)}(t) : \underline{n} \to \mathbb{R}$ then we can consider $\Delta x(t) \in \mathbb{R}^{n-1}$ as the discrete derivative of $x(t)$ with respect to the agent index i. Δ is thus not a differential but a difference operator. For abbreviation, we further define the *maximal gap* $\max \Delta x := \max_{i \in \underline{n}} \Delta x_i$. In our setting with ordered initial opinions, the function $x_{(\cdot)}(0)$ is monotonously increasing. Thus, its difference function $\Delta x_{(\cdot)}(0)$ is non-negative.

[c]Very small surviving clusters are neglected by this rule, but their existence is systematic as shown by the analysis of a rate equation for the density of opinions [13].

We are now able to define the *phone chain of closest* as a communication regime, which will later on guide us to a fair approximation of ε_{low}.

Definition 2 (Phone Chain of Closest). Let $n \in \mathbb{N}$ be the number of agents. A WD model of opinion dynamics is ruled by a *phone chain of closest* if the communicating agents at timestep $t \in \mathbb{N}$ are $(t \mod (n-1))+1$ and $(t \mod (n-1))+2$.

The phone chain of closest is $(1,2),(2,3),(3,4),\ldots,(n-1,n),(1,2)$, and so forth. This sequencing communication strategy provides a nice proof for the following proposition.

Proposition 1. *Let $x(0) \in \mathbb{R}^n$ be an ordered initial profile and let $\mu \in]0, 0.5]$. It holds that*

$$\max \Delta x(0) \le \varepsilon_{\text{low}} \le \max_{i \in \underline{n-1}} \sum_{j=0}^{i-1} \mu^j \Delta x_{i-j}(0). \tag{1}$$

For a proof, see Appendix A.1. Figure 1 shows how the phone chain of closest works.[d]

If we define $\text{range}(x) = \sum_{i \in \underline{n-1}} \Delta x_i = x_n - x_1$ and $\lceil \cdot \rceil$ as rounding a real value to the upper integer, then we can derive a corollary with a simpler, but not as sharp, bound.

Corollary 1. *If Proposition 1 holds, then it also holds that*

$$\varepsilon_{\text{low}} \le \frac{1 - \mu^{\lceil \frac{\text{range}(x(0))}{\max \Delta x(0)} \rceil}}{1-\mu} \max \Delta x(0). \tag{2}$$

For a proof, see Appendix A.2.

From Corollary 1, one can see that ε_{low} is determined mostly by the maximal gap, μ, and the ratio of the maximal gap and the difference between the two most extreme opinions in the initial profile. For $\mu = 0.5$, the estimate shows that enforcing consensus is always possible for ε which is twice the maximal gap of the initial profile.

Simulation-based studies often use initial profiles $x(0) \in [0,1]^n$ with random and uniformly distributed opinions. The length of the maximal gap in such a profile can be estimated by Whitworth's formula (3) and is thereby dependent on the number of agents:

$$P(\max \Delta x > \varepsilon) = \sum_{k=1}^{\lfloor \frac{1}{\varepsilon} \rfloor} (-1)^{k+1} (1 - k\varepsilon)^{n-1} \binom{n}{k}. \tag{3}$$

In terms of statistical theory, the formula is about the spacings in an order statistics of n independent uniformly distributed random variables [20]. Figure 2 shows the probability that the maximal gap is larger than ε with $\varepsilon \in [0,1]$.

[d]We suspect that our estimate is not strict, because we also studied regimes other than the phone chain of closest. However, the phone chain of closest delivers the best result we are able to prove analytically. Still, the question of the strictly lowest ε_{low} remains open.

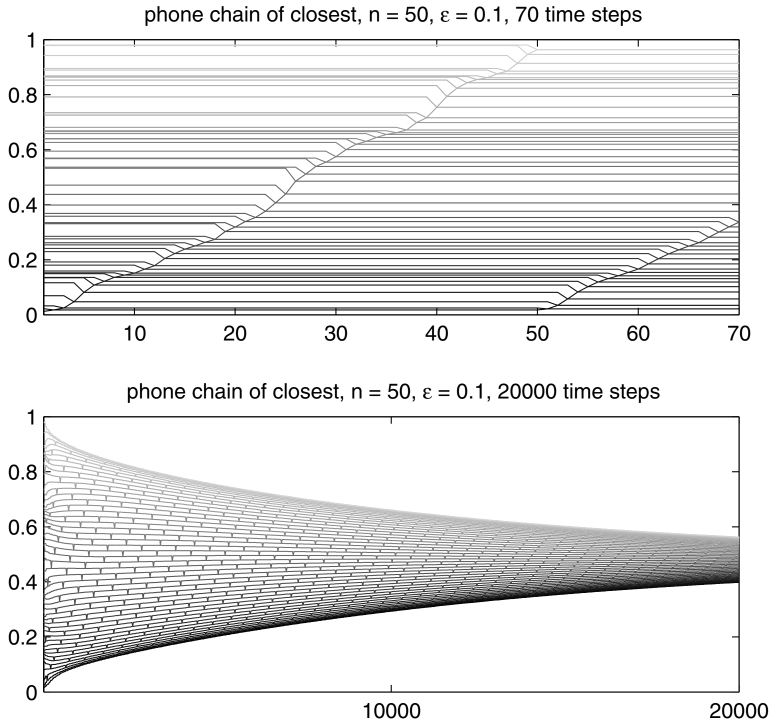

Fig. 1. The phone chain communication regime of closest.

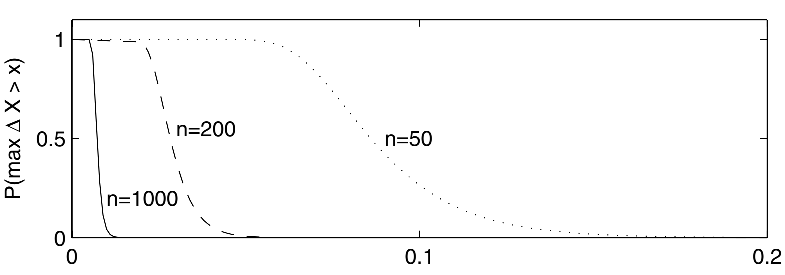

Fig. 2. $P(\max \Delta x > \varepsilon)$ for random equally distributed $x \in \mathbb{R}^n$ for some n.

Based on this distribution, it is possible to derive an estimate for the expected size of the maximal gap in an initial opinion profile. But there is an additional insight here: the larger the population, the smaller the expected size of the maximal gap. This leads to the conclusion that, for a very large number of agents who are equally distributed, consensus is possible for extremely low values of ε. If we assume that the maximal gap converges to zero as the number of agents increases, then it is possible to reach consensus for every ε with a sufficiently large number of agents. However, reasoning for infinitely many agents is not appropriate since every real society is finite in size. This finite size assumption is where many analytic approaches reach their limit.

We now ask the other way around: How large may ε be such that preventing consensus is still possible? At first we see that this question is not detailed enough to be interesting. Preventing consensus is obviously possible if we forbid one agent to communicate with all others, e.g. by underlying a disconnected social network. A better question is: How high may ε be such that preventing consensus is still possible, even if we switch at some timestep to an arbitrary communication regime? The biggest possible ε is called $\varepsilon_{\text{high}}$.

Proposition 2. *Let* $x(0) \in \mathbb{R}^n$ *be an ordered initial profile and let* $0 < \mu < 0.5$. *Then,*

$$\varepsilon_{\text{high}} = \max_{k \in \underline{n-1}} \left(\frac{1}{n-k} \sum_{i=k+1}^{n} x_i(0) - \frac{1}{k} \sum_{j=1}^{k} x_j(0) \right). \tag{4}$$

For a proof see Appendix A.3 and Lemma 1, which states that the mean opinion is conserved by the process of opinion formation.

If we regard random and uniformly distributed $x_i(0) \in [0, 1]$ for an n approaching infinity, then $\varepsilon_{\text{high}}$ is computed as the distance of the central points of two arbitrary disjoint intervals whose union is $[0, 1]$. Thus, $\varepsilon_{\text{high}} \to 0.5$ as $n \to \infty$. This proves Fortunato's universality result [11] by showing that preventing consensus is impossible for a large enough number of connected agents for $\varepsilon > 0.5$. Furthermore, Fortunato delivers evidence that consensus is not possible for $\varepsilon < 0.5$ as $n \to \infty$ and random pairwise communication regardless of an underlying network topology. Proposition 1 shows that there are specific communication orders that lead the society to consensus even for very low values of ε for every finite but arbitrarily large number of agents. However, the probability of obtaining one of these consensus-enforcing communication orders when picking it out of the set of random pairwise communication orders would probably approach zero in the limit of large n. Hence, if we are free to choose or to influence the communication order, then Fortunato's claim that consensus is not possible for $\varepsilon < 0.5$ is disproved. It remains to prove the impossibility of consensus for $\varepsilon < 0.5$ under random pairwise communication in the sense that our consensus-enforcing communication orders approach a probability of zero as n increases.

The interval $[\varepsilon_{\text{low}}, \varepsilon_{\text{high}}]$ is the range where both enforcing and preventing consensus is possible with an appropriate communication regime. Both bounds of the interval depend on the initial profile $x(0)$. Figure 3 provides some numerical evidence about the possibilities that can be reached with manipulation of the communication regime. The data in line one come from Fig. 4 in Ref. 4 (visually extracted) with 250 simulation runs with random and equally distributed initial profiles and $n = 1000$. For line two, we took 250 randomly chosen and equally distributed profiles with $n = 1000$ and show the maximal ε_{low} and minimal $\varepsilon_{\text{high}}$ that occurred in all 250 profiles (all computed with Propositions 1 and 2). Enforcing and preventing

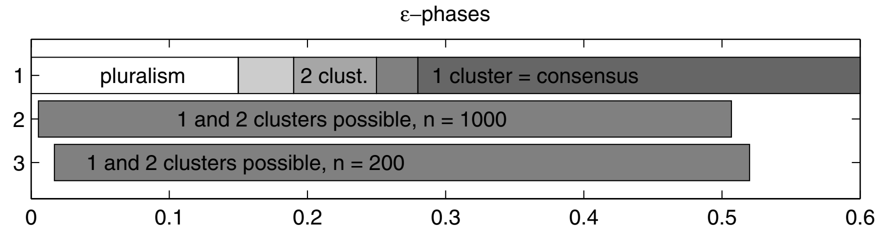

Fig. 3. Numerical $[\varepsilon_{\mathrm{low}}, \varepsilon_{\mathrm{high}}]$ (second darkest gray) where consensus and polarization occurs (1) and is possible (2 and 3).

consensus was possible in the displayed interval for all 250 selected profiles. The same is true in line three for $n = 200$.

From this figure, we can also see that for finite populations and random initial profiles avoiding consensus is possible even for $\varepsilon > 0.5$ (in contrast to Fortunato). The larger the population, the smaller ε can be while still guaranteeing the possibility of consensus. The larger the population, the smaller the upper limit of ε for which consensus can be prevented is.

4. Individual Strategies that Increase Chances for Consensus

In the previous section, we applied a mechanism for enforcing consensus that was built on knowledge about all people's opinions. We now want to leave behind this idea of global knowledge and the great master plan for communication and go to agent-based strategies, which may also promote consensus. Our agents do not know the opinions of all other agents and thus do not know if they are in the center or at the extremes of the opinion space. The agents follow rules that only require knowledge of their own communication history.

4.1. *Balancing and curious agents with directions*

From the huge set of possible individual communication strategies, we focus on balancing and curious agents. Consider an agent who has communicated with another agent and adapted his opinion accordingly. A *balancing agent* will now search for an agent whose opinion is contrary to that of the previous communication partner, while ignoring all other agents. A *curious agent* will instead seek out a new communication partner whose opinion is in line with that of the previous communication partner, again ignoring other agents.

To prevent agents from not finding an agent to compromise with, we introduce a new parameter, $f_{\max}$, which represents a maximal level of frustration. Specifically, it is the number of unsuccessful attempts an agent sticks to the rule before abandoning it. Thus, agents are not forced to follow the strategies forever. We store relative opinions of potential communication partners, more precisely, the direction, and individual frustration levels for all agents in a vector $d \in \mathbb{Z}^n$. If d_i is negative, then agent i wants to compromise with an agent with a lower opinion. If d_i is positive, then agent i wants to compromise with an agent with a higher opinion. If d_i is zero agent i has no preferred direction. The absolute values of the directions represent the frustration level. The lower the absolute value, the higher the frustration is; if it reaches zero, then the agents no longer care about the direction of potential communication partners.

Frustration and direction are additional factors that affect an agent's tendency to compromise. Agents i and j only compromise if both agents' opinions are in the corresponding set of opinions the other agent looks for, i.e. $d_i * d_j \leq 0$. If they are not in the set, but are closer than ϵ to each other, then they both reduce the absolute value of their frustration levels each by one point. Thus, the absolute values of d_i and d_j decrease. After a successful compromise, agents set d_i to $f_{\max}$ with the sign indicating the new search direction. Curious agents differ from balancing agents in the sign of d_i. Besides this restriction, we return to random pairwise communication. The corresponding pseudo-code can be found in Appendix B.

Societies of balancing and curious agents are essentially identical in their dynamics if $\mu = 0.5$. It is interesting to note that after a compromise between two balancing agents, we end up with two agents with the same opinion searching in opposite directions; however, the same applies to curious agents but with agents whose indices are reversed. Thus, clustering outcomes are identical for both balancing and curious agents when $\mu = 0.5$.

4.2. *Simulation setup*

For both strategies, balancing and curious agents, we ran simulations for the values $\mu = 0.2, 0.5$, $n = 50, 100, 200$, $\varepsilon = 0, \overset{+0.01}{\dots}, 0.35$, and $f_{\max} = 0, 1, 2, 4, 8, 16, 32$. For each point in this parameter space, we have 3000 independent simulation runs with random initial profiles and random selection of communication partners.[e]

Each simulation run stops when we reach a configuration where all indirectly connected[f] subgroups of agents have a maximal opinion difference smaller than ε and thus cannot further split. The mean preserving property (see Lemma 1 in Appendix A.3) of the dynamics permits a calculation of the long term limit of the convergence process. We consider the *average size of the biggest cluster* after stabilization as a measure for the possibility of consensus.[g] All simulations were implemented using ANSI-C. The program code is available on request from the second author.

[e] To check larger numbers of agents, we performed 3000 independent simulation runs for balancing agents for $n = 500, 1000$, $\varepsilon = 0, \overset{+0.01}{\dots}, 0.35$, $\mu = 0.2, 0.5$, and $f_{\max} = 0, 1, 2, 4, 8$.

[f] Two agents are connected if their opinions differ by not more than ε. They are indirectly connected if there is a chain of connected agents between them.

[g] Another possible measure would be the average number of clusters. But the Weisbuch and Deffuant model is known to produce minor clusters of only a few agents [13, 21].

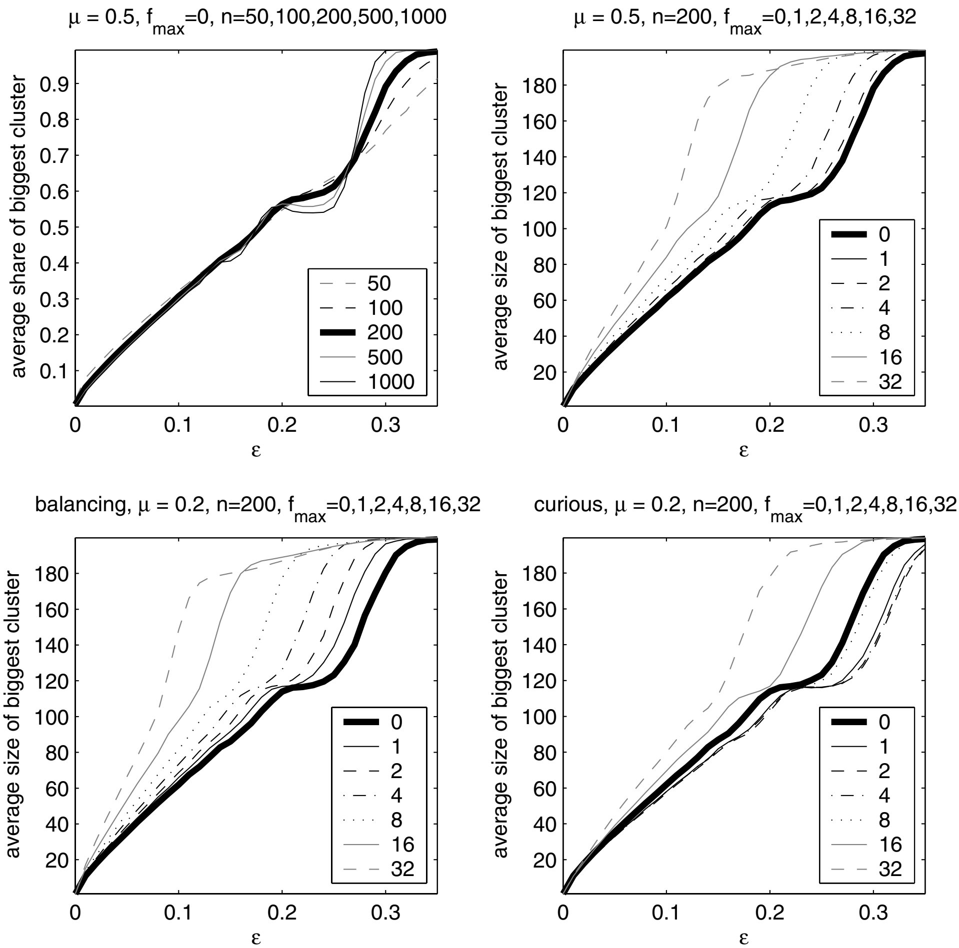

Fig. 4. The average size of the biggest cluster for initial profiles with 50, 100, 200, 500 and 1000 agents (top-left) and $\mu = 0.5$, for 200 balancing/curious agents with $\mu = 0.5$ (top-right), for 200 balancing agents with $\mu = 0.2$ (bottom-left), and 200 curious agents with $\mu = 0.2$ (bottom-right).

4.3. *Simulation results*

Figure 4 shows the results for the average size of the biggest cluster. The thick line always represents the average size of the biggest cluster for $f_{\max} = 0$ and $n = 200$. The upper-left plot shows how this line changes for varying numbers of agents. The plateau at $\varepsilon = 0.2$ shows the characteristic polarization phase where agents form two big clusters (see, for instance, Ref. 4). We see that this plateau becomes more pronounced for larger n and less distinct for smaller n.[h]

[h]The small "hill" at $\varepsilon = 0.19$ for $n = 500, 1000$ is another interesting phenomenon related to the measure of the average size of the biggest cluster, but it is beyond the scope of this paper.

Next, we examine how the transition from polarization to consensus shifts to other ε-regions when agents are balancing or curious. We concentrate our discussion on the case of 200 agents, but we checked that the shifts for $n = 50, 100, 500, 1000$ are similar.[i]

The upper-right plot in Fig. 4 shows the effect of balancing and curious agents for $\mu = 0.5$, in which case these two types exhibit the same dynamics. In the lower plots, we distinguish between balancing and curious for agents who are more cautious, i.e. $\mu = 0.2$. The thin lines show the effects of an increase in the maximal frustration $f_{\max}$ under a given strategy and μ. The main conclusions from Fig. 4 are: Being balancing has a positive effect on the chances for consensus. For $\mu = 0.5$, this holds for all maximal frustrations $f_{\max} > 1$. The same holds trivially for curious agents under $\mu = 0.5$. A smaller μ, which means being more cautious, supports the positive effects for balancing agents. However, a smaller μ does not support the positive effects of curious agents.[j]

Figure 4 is based on aggregated data. To give a more detailed picture of the dynamics, Figs. 5 and 6 show some single simulation runs. Figure 5 shows how balancing agents with an intermediate frustration maximum are positively effected in finding a consensus by being more cautious. Figure 6 shows the ambivalent effects of curious agents who are cautious. While a high frustration maximum can foster consensus, an intermediate frustration maximum may even prevent consensus. We see that almost every curious agent has to cross the central opinion if curious agents want to reach a consensus. Figures 5 and 6 also demonstrate the fact that consensus due to balancing or curious agents is paid by longer convergence time.

5. Discussion

Our analytical results describe the possibilities of consensus in the Weisbuch and Deffuant model and we prove the universality of the consensus threshold in the sense of Fortunato [11]. Both enforcing and preventing consensus is possible in a large interval for values of ε, and we give an impression of how it scales with the number of agents and the cautiousness parameter. This shows the large impact that the control of communication has on consensus formation in the Weisbuch and Deffuant model in finite populations. Communication control is a feature of real opinion dynamics, which is to some extent manipulable through organizations. Therefore, our results are of interest for those who aim at designing communication and discussion processes and want to foster consensus or dissent (see, for example, Ref. 22).

[i]Data for $n = 50, 100, 500, 1000$ are available on request from the second author.

[j]An interesting but small effect is that the general tendency of an increase in chances of consensus with an increase in $f_{\max}$ is sometimes slightly violated. For instance, for balancing agents with $\mu = 0.2$ and $f_{\max} = 16$, we observe a slightly larger average size of the biggest cluster than for $f_{\max} = 32$. We suspect that this is a systematic effect and not caused by chance. However, the effect is so small that we did not study the causes further.

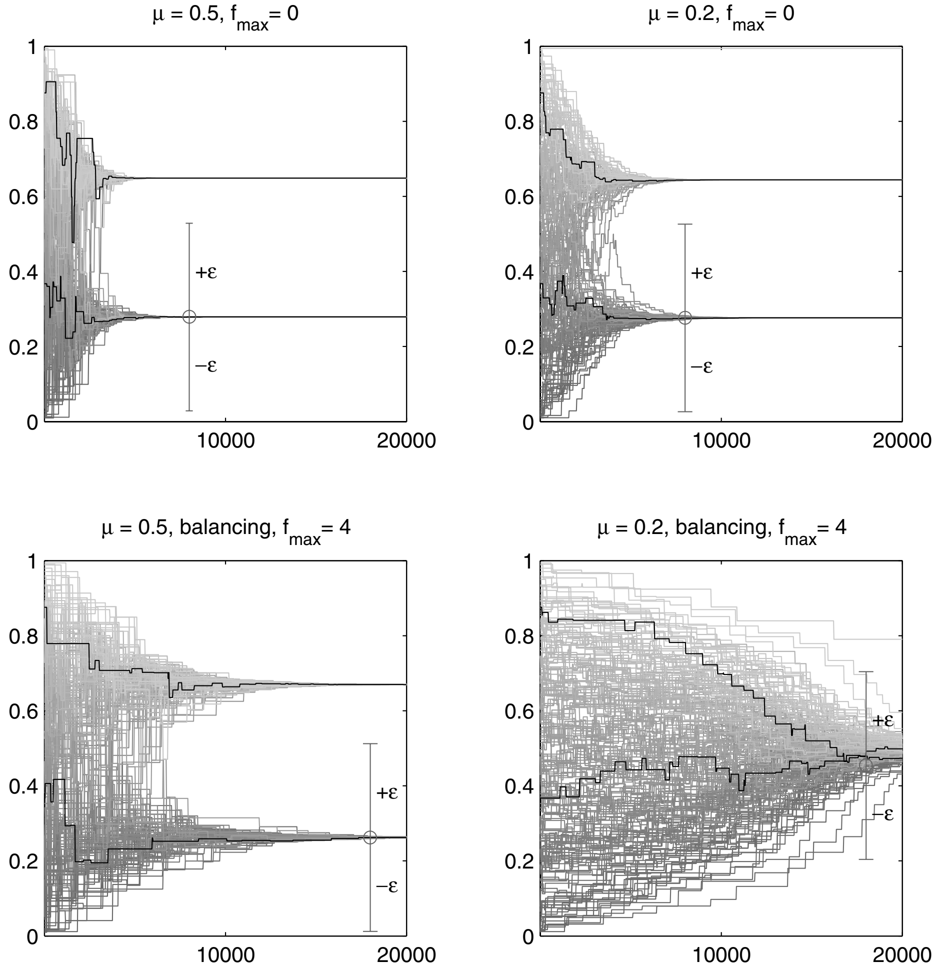

Fig. 5. Example processes for balancing agents with $n = 200$ and $\varepsilon = 0.25$. This demonstrates that cautiousness fosters the positive effect of balancing.

Generally, continuous opinion dynamics under bounded confidence is driven by the opposition of the consensus-promoting force of averaging and the separating force of bounded confidence. Dynamics start at the extremes of the opinion space. Specifically, the most extreme agents move towards less extreme positions and thus higher densities of opinions evolve at both extremes. These two high density regions attract agents from the center and may lead to a split in the opinion range.

We explored by simulation the dynamics of societies where each individual behaves as "balancing" or "curious." Balancing agents tend to move in a narrow zigzag around their first opinion, while curious agents tend to move in a wide zigzag exploring almost the whole opinion space. Therefore, both strategies have a tendency to prevent a rapid clustering. Balancing agents do this by seeking input from

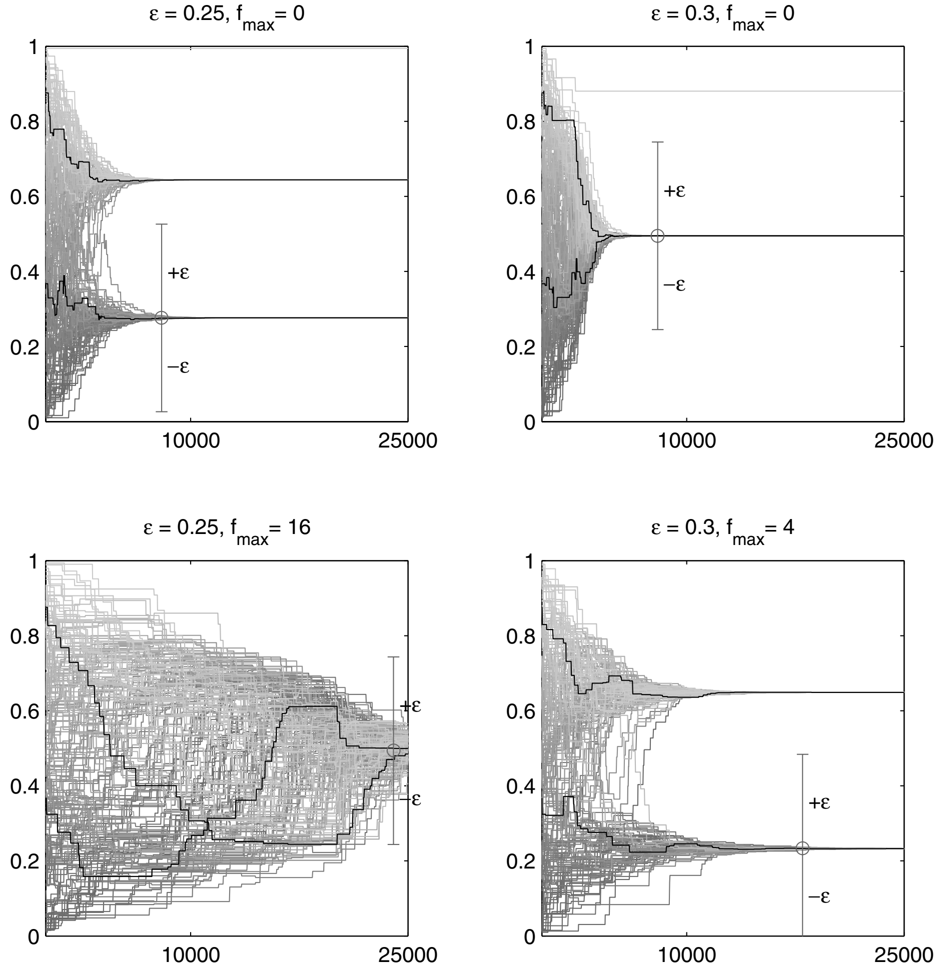

Fig. 6. Example processes for curious agents with $n = 200$ and $\mu = 0.2$. This demonstrates that for cautious agents being curious with a high frustration level one can foster consensus (left-hand side) but for a low frustration level it may destroy consensus (right-hand side).

both sides, which prevents them from quickly being absorbed by a nearby cluster. Curious agents tend to run through and finally break out of a cluster they recently joined. They tend to explore more of the opinion space. Since only the clustering is slowed down, while the overall contraction process of the opinion profile maintains its speed, the chances for consensus are increased.

We further extend the analyses in Refs. 9 and 19 on the role of the cautiousness parameter μ. Particularly in the first part of this paper, we see that cautiousness significantly controls the possibility of consensus. Furthermore, the second part of the paper illustrates the intriguing interplay of this parameter with agents' communication strategies.

To summarize, if you want your agents to foster consensus by balancing, you should appeal to them to be cautious. If you want them to foster consensus by being curious, you should appeal to them not to be cautious, otherwise you may even get a negative effect when agents have a low frustration maximum. This results from the fact that balancing agents prevent clustering by trying to avoid early absorption into clusters and thus smaller steps, i.e. smaller μ has a positive effect on formation of consensus. Curious agents prevent clustering by getting out of clusters they recently entered; hence, smaller steps have a negative effect. In general, the impact of being balancing is higher than that of being curious, yet both individual strategies can foster consensus.

Appendix A. Appendix for Proofs

A.1. *Proof of Proposition 1*

Proof. The left inequality results from the fact that an $\varepsilon < \max_{i\in\underline{n-1}} \Delta x_i(0)$ can obviously not bridge the maximal gap; thus, the opinion profile will be divided into the two groups above and below this gap forever, regardless of any communication structure.

To show the right inequality, let $\varepsilon > \max_{i\in\underline{n-1}} \sum_{j=0}^{i-1} \mu^j \Delta x_{i-j}(0)$. We will show that our specific communication regime, the phone chain of closest, drives the dynamic to a consensus.

First, the phone chain of closest cannot change the order of the opinion profile. Thus, it holds for all $t \in \mathbb{N}_0$ that $x_1(t) \leq \cdots \leq x_n(t)$.

In a first step, we will look at the $n-1$ first timesteps, thus the first phone chain round. After one round, we will see that the maximal gap in $\Delta x(n-1)$ has shrunk substantially, and we can conclude with an inductive argument.

Let us consider that there is no bounded confidence restriction by ε, thus in every timestep two opinions really change (if they are not already equal). We will derive equations for Δx in the timesteps $1, \ldots, n-1$ under this assumption. After that, we will see that ε does not restrict this dynamic.

Let $i \in \underline{n-1}$ be an arbitrary agent. We focus on Δx_i, the gap between i and $i+1$, for all timesteps and deduce formulas only containing values of the initial profile. Agent i at timestep $i-1$ has communicated recently with agent $i-1$ and will communicate with agent $i+1$. Thus

$$\Delta x_i(i-2) = \cdots = \Delta x_i(1) = \Delta x_i(0). \tag{A.1}$$

Due to the communication with agent $i-1$, agent i moves towards $i-1$ thus Δx_i gets larger:

$$\Delta x_i(i-1) = \Delta x_i(i-2) + \mu \Delta x_{i-1}(i-2). \tag{A.2}$$

By recursion of (A.1) and (A.2), it follows that

$$\begin{aligned}\Delta x_i(i-1) &= \Delta x_i(0) + \mu\Delta x_{i-1}(0) + \cdots + \mu^{i-2}\Delta x_2(0) + \mu^{i-1}\Delta x_1(0)\\ &= \sum_{j=0}^{i-1} \mu^j \Delta x_{i-j}(0). \end{aligned}\tag{A.3}$$

Going one step further to the communication of i and $i+1$, where their opinion gets closer, we get the following:

$$\Delta x_i(i) = \Delta x_i(i-1) - 2\mu\Delta x_i(i-1). \tag{A.4}$$

We use $\Delta x_i(i-1)$ as an abbreviation for the right-hand side of (A.3) which only contains expressions at timestep 0.

In the next step, Δx_i becomes larger as agent $i+1$ moves towards agent $i+2$:

$$\begin{aligned}\Delta x_i(i+1) &= \Delta x_i(i) + \mu\Delta x_{i+1}(i)\\ &\overset{(A.4)(A.2)}{=} (1-2\mu)\Delta x_i(i-1) + \cdots + \mu(\Delta x_{i+1}(i-1) + \mu\Delta x_i(i-1))\\ &\overset{(A.1)}{=} \mu\Delta x_{i+1}(0) + (1-2\mu+\mu^2)\Delta x_i(i-1). \end{aligned}\tag{A.5}$$

To complete all timesteps until $t = n-1$, we have to mention

$$\Delta x_i(i+1) = \Delta x_i(i+2) = \cdots = \Delta x_i(n-1). \tag{A.6}$$

For Δx_{n-1}, there is no Eq. (A.5); the last value after the phone chain round is computed by Eq. (A.4).

To make all these equations valid and thus to ensure that no opinion change is prevented by ε, it must hold for all $i \in \underline{n-1}$ that $\Delta X_i(i-1) < \varepsilon$. Looking at (A.3), we see that this is the case by construction of the lower bound of ε.

From Eqs. (A.3), (A.5) and (A.6), we get

$$\begin{aligned}\Delta x_i(n-1) &= \mu\Delta x_{i+1}(0) + (1-2\mu+\mu^2)\sum_{j=0}^{i-1}\mu^j\Delta x_{i-j}(0)\\ &\le \left(\mu + (1-2\mu+\mu^2)\sum_{j=0}^{i-1}\mu^j\right)\max\Delta x(0)\\ &= \left(\mu + (1-\mu)^2\frac{(1-\mu^i)}{1-\mu}\right)\max\Delta x(0)\\ &= (1-\mu^i+\mu^{i+1})\max\Delta x(0). \end{aligned}\tag{A.7}$$

Thus, it holds that $\max\Delta x(n-1) \le (1-\mu^i+\mu^{i+1})\max\Delta x(0)$. It is easy to see that $k := 1-\mu^i+\mu^{i+1} < 1$ for $0<\mu<1$.

For the next phone chain rounds, we can conclude with the same procedure and it will hold that $\max\Delta x(t(n-1)) \le k^t \max\Delta x(0)$. Thus, $\max\Delta x(t)$ converges to zero, which implies that the process converges to a consensus. □

A.2. *Proof of Corollary 1*

Proof. With abbreviation $x := x(0)$, we use the equations $\mathrm{range}(x) = \sum_{i=1}^{n-1} \Delta x_i$ to derive $\mathrm{range}(x) \leq \lceil \frac{\mathrm{range}(x)}{\max \Delta(x)} \rceil \max \Delta x$. This gives

$$\sum_{i=1}^{n-1} \Delta x_i \leq \sum_{i=1}^{\lceil \frac{\mathrm{range}(x)}{\max \Delta(x)} \rceil} \max \Delta(x) \tag{A.8}$$

and therefore

$$\varepsilon_{\mathrm{low}} \leq \max_{i \in \underline{n-1}} \sum_{j=0}^{i-1} \mu^j \Delta x_{i-j} \leq \sum_{i=0}^{\lceil \frac{\mathrm{range} x}{\max \Delta(x)} \rceil} \mu^{i-1} \max \Delta(x). \tag{A.9}$$

Taking the right-hand side of (A.8) and transforming it to $\frac{1-\mu^{\lceil \frac{\mathrm{range}(x)}{\max \Delta x} \rceil}}{1-\mu} \max \Delta x$ finally provides us with Corollary 1. $\square$

A.3. *Proof of Proposition 2*

Lemma 1. *Let $x(0) \in \mathbb{R}^n$ be an initial profile and $(x(t))_{t \in \mathbb{N}_0}$ be a process in a WD model of opinion dynamics with arbitrary ε, μ. For every timestep $t \in \mathbb{N}_0$, it holds that*

$$\frac{1}{n} \sum_{i=1}^{n} x_i(t) = \frac{1}{n} \sum_{i=1}^{n} x_i(0). \tag{A.10}$$

Proof. Obvious by Definition 1. $\square$

Proof of Proposition 2. Let $\varepsilon \leq \varepsilon_{\mathrm{high}}$. Let us divide the set of agents according to the maximal k in Eq. (4) into two subsets $I_1 = \{1, \ldots, k\}, I_2 = \{k+1, \ldots, n\}$. We choose a communication regime where both subgroups find their respective consensus $x_1 = \cdots = x_k = c_1, x_{k+1} = \cdots = x_n = c_2$. This should be possible, otherwise we are have established a persistent dissence. Due to Lemma 1 and Eq. (4), it holds that $|c_1 - c_2| \geq \varepsilon$ and communication is no longer possible between the subgroups. $\square$

B. Appendix for Pseudo Code

For balancing agents, we use:

```
initialize X[]
initialize D[] = (0,0,...,0)
WHILE not clustered(X) AND changes possible
   choose agent i,j
   IF |X[i]-X[j]| <= epsilon
```

```
 6:         IF (X[j]-X[i])*D[i] >= 0 AND (X[j]-X[i])*D[i] >= 0
 7:             X[i]=X[i] - mu*(X[i]-X[j])
 8:             X[j]=X[j] + mu*(X[i]-X[j])
 9:             D[i] = + sign(X[i]-X[j]) * fmax
10:             D[j] = - sign(X[i]-X[j]) * fmax
11:         ELSE
12:             IF D[i]!=0
13:                 D[i]= D[i] - sign(D[i])
14:             ENDIF
15:             IF D[j]!=0
16:                 D[j]= D[j] - sign(D[j])
17:             ENDIF
18:         ENDIF
19:     ENDIF
20: ENDWHILE
```

For curious agents, we use same code with signs switched in lines nine and ten.

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