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A stabilization theorem for dynamics of continuous opinions

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Abstract

A stabilization theorem for processes of opinion dynamics is presented. The theorem is applicable to a wide class of models of continuous opinion dynamics based on averaging (like the models of Hegselmann–Krause and Weisbuch–Deffuant). The analysis detects self-confidence as a driving force of stabilization.

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1. Modeling of opinion dynamics

Consider a group of n agents each having an opinion about a certain issue. The agents may revise their opinions according to the opinions of other agents. If revising goes on, we have a process of opinion formation. The understanding of phenomena like stabilization of opinions distribution, finding a consensus, polarization into opinion clusters, extremism or spreading of minority opinions is of interest in sociology, political science and economics (e.g., price setting or customer's opinions about brands). Mathematical models and their analysis should detect driving forces of opinion dynamics.

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Here, we consider the continuous opinion approach where the opinion space is a real interval, see Refs. [1–5]. Thus, the opinion dynamic can be driven by compromising.

For the model of continuous opinion dynamics we consider $\underline{n} := \{1, \dots, n\}$ agents who discuss their opinions. We call $X(t) \in \mathbb{R}^n$ an *opinion profile* at time step $t \in \mathbb{N}_0$, where $X_i(t)$ represents the opinion of agent i.

Definition 1 (*confidence matrix*). Let $X(t)$ be an opinion profile at time step $t \in \mathbb{N}_0$. A matrix $A(X(t), t) \in \mathbb{R}^{n\times n}_{\geqslant 0}$ is called *confidence matrix* if it is row-stochastic.

The entry $A(X(t), t)_{[i,\, j]}$ represents the weight (or confidence) that agent i distributes to the opinion of agent j. Notice that the confidence matrix is a function of the actual opinion profile and of the specific time step. Let $X(0) \in \mathbb{R}^n$ be a starting opinion profile. The *process of continuous opinion dynamics* is the series of opinion profiles $(X(t))_{t\in\mathbb{N}_0}$ recursively defined through

$$X(t+1) = A(X(t), t)X(t)\ .$$

Thus, each new opinion is a weighted arithmetic mean of all the old opinions. It holds $X(t+1) = A(X(t), t) \cdots A(X(0), 0)X(0)$ by iteration.

This very general agent-based setting gets explicit by defining how the confidence matrix is constructed. The setting also contains models with heterogeneous agents, underlying network structures and various updating rules, as long as repeated averaging drives the dynamic. Further on m-dimensional opinions can be modelled by regarding $X(t) \in \mathbb{R}^{n\times m}$.

DeGroot [1] analyzes the model for fixed A and gives conditions for reaching consensus. Chatterjee and Seneta [2] derived some generalizations for $A(t)$ in the sense of hardening of positions.

In this paper we want to treat the much more complicated profile dependent case, where no analytical results are available. We will point out weak but sufficient conditions on the confidence matrices such that the process converges to a fixed opinion profile. But these conditions are not necessary. These conditions are for all $t \in \mathbb{N}_0$

(1) *Every agent got a little bit of self-confidence.* The diagonal of $A(X(t), t)$ is positive. For every agent $i \in \underline{n}$ it holds $a_{ii} > 0$.
(2) *Confidence is mutual.* Zero-entries in $A(X(t), t)$ are symmetric. For every two agents $i, j \in \underline{n}$ it holds $a_{ij} > 0 \Leftrightarrow a_{ji} > 0$.
(3) *Positive weights do not converge to zero.* There is $\delta > 0$ such that the lowest positive entry of $A(X(t), t)$ is greater than δ.

In the bounded confidence model of Hegselmann–Krause [3] the confidence matrix is defined for $\varepsilon > 0$ and an opinion profile $X \in \mathbb{R}^n$ as

$$A(X)_{ij} := \begin{cases} \dfrac{1}{|I(i, X)|} & \text{if } j \in I(i, X) := \{j \in \underline{n} \mid |X_i - X_j| \leqslant \varepsilon\}\ , \\ 0 & \text{otherwise}\ , \end{cases}$$

In the basic model of Weisbuch et al. [5] two randomly chosen agents $i, j \in \underline{n}$ interact in each time step. They adjust their opinions if $|X(t)_i - X(t)_j| \leqslant \varepsilon$ by a step of $\mu|X(t)_i - X(t)_j|$ towards each other ($0 < \mu < 0.5$). Thus a confidence matrix in one time step is the unit matrix besides the entries $a_{ii} = a_{jj} = 1 - \mu$ and $a_{ij} = a_{ji} = \mu$.

Thus, it is easy to check that both the Hegselmann–Krause and the basic Weisbuch–Deffuant model fulfill conditions (1)–(3).

2. The stabilization theorem

For abbreviation, we define for a series of matrices $(A(t))_{t \in \mathbb{N}_0}$ the *accumulation* from time step t_0 to t_1 as $A(t_0, t_1) := A(t_1 - 1)A(t_1 - 2) \cdots A(t_0 + 1)A(t_0)$. A *consensus matrix* should be a row-stochastic matrix with equal rows. With definition $A(t) := A(X(t), t)$ we can write $X(t) = A(0, t)X(0)$. We will show that $\lim_{t\to\infty} A(0, t)$ converges to a constant matrix. This implies that $(X(t))_{t \in \mathbb{N}_0}$ converges to a constant opinion profile.

Theorem 2. *Let $(A(t))_{t \in \mathbb{N}_0} \in \mathbb{R}^{n \times n}_{\geqslant 0}$ be a series of confidence matrices. If each matrix fulfills properties* (1)–(3), *there exists a time step t_0 and pairwise disjoint classes of agents $\mathscr{I}_1 \cup \cdots \cup \mathscr{I}_p = \underline{n}$ such that*

$$\lim_{t\to\infty} A(0, t) = \begin{bmatrix} K_1 & & 0 \\ & \ddots & \\ 0 & & K_p \end{bmatrix} A(0, t_0)$$

and $K_1, \ldots, K_p$ are quadratic consensus matrices in the sizes of $\mathscr{I}_1, \ldots, \mathscr{I}_p$. (For the block structure we must sort matrix indices according to $\mathscr{I}_1, \ldots, \mathscr{I}_p$.)

In front of the proof some explanations and necessary propositions: If we multiply the consensus matrix K_i with an arbitrary vector then we get a vector with all entries equal. Thus, the theorem says that every starting opinion profile develops to a time step t_0, where the agents split into some independent classes. The opinions of the agents in these classes converge to consensus.

For a matrix $A \in \mathbb{R}^{n \times n}_{\geqslant 0}$ we say that two agents $i, j \in \underline{n}$ *communicate*, if there exist agents $i = i_1, \ldots, i_k = j \in \underline{n}$ such that for all $l = 1, \ldots, k - 1$ the agents i_l and $i_l + 1$ trust each other ($a_{i_l i_{l+1}} > 0$). It is easy to see that the set of agents $\underline{n}$ divides for every opinion profile into self-communicating classes $\mathscr{I}_1, \ldots, \mathscr{I}_p$. This means, each agent communicates with every other agent in his class, but with no agent outside. Notice that the structure of self-communicating classes of indices depends only on the zero-pattern of the matrix.

We need the following three propositions to prove the theorem.

Proposition 3. *Let $(A(t))_{t \in \mathbb{N}_0} \in \mathbb{R}^{n \times n}_{\geqslant 0}$ be a series of matrices fulfilling condition* (1), *then there exists a series of time steps $t_0 < t_1 < t_2 < \cdots$ such that $A(t_0, t_1), A(t_1, t_2), \ldots$ got the same zero-pattern. Let $\mathscr{I}_1, \ldots, \mathscr{I}_p$ be the self-communicating classes of agents of the matrices $A(t_0, t_1), A(t_1, t_2), \ldots$. If we sort the agents of every matrix by simultaneous* *row and column permutations, then we got a block matrix with strictly positive blocks on the diagonal* ($A(t_k, t_{k+1})_{[\mathcal{I}_i,\mathcal{I}_i]} > 0$ *for all* $k \in \mathbb{N}_0$, $i \in \underline{p}$) *and zero-blocks at all other positions* ($A(t_k, t_{k+1})_{[\mathcal{I}_i,\mathcal{I}_j]} = 0$ *for all* $k \in \mathbb{N}_0$ *and* $i,j \in \underline{p}$, $i \neq j$).

Proposition 4. *Let* $(A(t))_{t\in\mathbb{N}_0} \in \mathbb{R}^{n\times n}_{\geqslant 0}$ *be a series of confidence matrices fulfilling conditions* (1)–(3). *Then it holds for every two time steps* $t_0 < t_1$ *that the lowest positive entry of* $A(t_0, t_1)$ *is greater than* δ^{n^2-n+2}.

Proposition 5. *Let* $(A(t))_{t\in\mathbb{N}_0} \in \mathbb{R}^{n\times n}_{\geqslant 0}$ *be a series of row-stochastic matrices and let* $\delta_t > 0$ *be a series with* $\sum_{t=0}^{\infty} \delta_t = +\infty$. *If it holds for all* $t \in \mathbb{N}_0$ *that* $\min_{i,j} \sum_{k=1}^{n} \min\{a(t)_{ik}, a(t)_{jk}\} \geqslant \delta_t$ *then there exists a consensus matrix K such that* $\lim_{t\to\infty} A(0,t) = K$.

Proofs are in the appendix.

Proof. (of Theorem 2) Proposition 3 gives us time steps $t_0 < t_1 < t_2 < \cdots$ and classes of indices $\mathcal{I}_1, \ldots, \mathcal{I}_p$ such that each matrix $A(t_i, t_{i+1})$ got positive blocks $A(t_i, t_{i+1})_{[\mathcal{I}_j,\mathcal{I}_j]}$ for all $j \in \underline{p}$ and zeros elsewhere.

From Proposition 4 and condition (2) we can derive that the lowest entry of each $A(t_i, t_{i+1})_{[\mathcal{I}_j,\mathcal{I}_j]}$ is greater than δ^{n^2-n+2}. For the series $(A(t_i, t_{i+1})_{[\mathcal{I}_j,\mathcal{I}_j]})_{i\in\mathbb{N}_0}$ the assumptions of Proposition 5 are fulfilled for all $j \in \underline{p}$.

Further on it holds that $[\ldots A(t_1, t_2)A(t_0, t_1)]_{[\mathcal{I}_j,\mathcal{I}_j]} = \ldots A(t_1, t_2)_{[\mathcal{I}_j,\mathcal{I}_j]} A(t_0, t_1)_{[\mathcal{I}_j,\mathcal{I}_j]}$ due to the block structure. Thus there is a consensus matrix K_j such that $\lim_{i\to\infty} A(t_0, t_i) = K_j$. □

Thus, the convergence to an opinion profile with consensus subgroups is proved for the model of Hegselmann–Krause, where it was only proved for the one-dimensional case with no generalizations, and for the basic Weisbuch–Deffuant, which was only observed in simulation. Ben-Naim et al. [4] propose a differential equation for the opinion distribution for the basic Weisbuch–Deffuant model and have other arguments for stabilization. But they treat idealized $+\infty$ agents. We focus on the dynamic of a finite number of agents using completely different technics and generalizing to various models.

The theorem more colloquial: A process of continuous opinion dynamics stabilizes when (1) each agent has a little bit of self confidence, (2) confidence is mutual and (3) these two conditions do not fade away by convergence to zeros. This detects self-confidence as a driving force of stabilization in continuous opinion dynamics in a completely analytical way. If we had no self-confidence periodic behavior may happen. If we drop mutual confidence we might imagine an open-minded agent between two narrow-minded agents (the open-minded trusts the narrow-minded but they do not trust him). The open-minded may hop around in the space between but will not be converging.

The theorem secures stabilization for simulation of further models basing on averaging and fulfilling properties (1)–(3) which may contain multidimensional opinions, heterogeneous agents, network structures and sophisticated updating rules.

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Appendix A

A.1. Proof of Proposition 3

Notice that for any two non-negative matrices with positive diagonals $A, B \in \mathbb{R}^{m\times m}_{\geqslant 0}$ it holds that every entry which is positive in A or in B is also positive in AB. Therefore, more and more positive entries appear in $A(0,t)$ monotonously increasing with t. Thus, once there will be a time step t_0^* in which the maximum number of positive entries in $A(0,t)$ for all $t \in \mathbb{N}$ is reached. And it is clear that no matrix $A(t)$ with $t \geqslant t_0^*$ got a positive entry, where $A(0,t_0^*)$ has got a zero-entry.

If we look at the series $(A(t))_{t\geqslant t_0^*}$, we find another time step t_1^*, such that $A(t_0^*, t_1^*)$ has reached again the maximum number of positive entries, but there are less or equal positive entries as in $A(0,t_0^*)$.

If we continue like this we get a series $A((t_i^*, t_{i+1}^*))_{i\in\mathbb{N}_0}$ of accumulations in which positive entries vanish monotonously. Thus, once there will be a time step $t_k^* =: t_0$ for which the minimum of positive entries is reached and so with $t_i := t_{i+k}^*$ we got the asserted series of time steps.

For proving the block structure, we first notice that it is clear (due to the definition of self-communicating classes) that $A(t_k, t_{k+1})_{ij} = 0$ for all $i,j \in \underline{n}$ coming from different self-communicating classes. The last thing to show is, that for every self-communicating class $\mathscr{I}_l$ it holds that $A(t_k, t_{k+1})_{[\mathscr{I}_l,\mathscr{I}_l]}$ is strictly positive. For all $k \in \mathbb{N}_0, l \in \underline{p}$ the matrix $A(t_k, t_{k+1})_{[\mathscr{I}_l,\mathscr{I}_l]}$ is primitive (that means that one power is positive) because all agents are communicating and the diagonal is positive. The primitivity property depends only on the zero pattern of a matrix, which is equal in $A(t_k, t_{k+1})_{[\mathscr{I}_l,\mathscr{I}_l]}$ for every $k \in \mathbb{N}$. Thus, there exists $z \in \mathbb{N}$ such that $A(t_0, t_z)_{[\mathscr{I}_l,\mathscr{I}_l]} = A(t_{z-1}, t_z)_{[\mathscr{I}_l,\mathscr{I}_l]} \cdots A(t_0, t_1)_{[\mathscr{I}_l,\mathscr{I}_l]}$ is strictly positive. Thus, $A(t_k, t_{k+1})_{[\mathscr{I}_l,\mathscr{I}_l]}$ must be strictly positive for all k because otherwise, there were less positive entries than in later accumulations, which is a contradiction to the minimality of positive entries proved before.

A.2. Proof of Proposition 4

Let $t_0 < t_1$ and $n^* := t_1 - t_0$. Let $\mu(A)$ be the lowest positive entry of the non-negative matrix A. With condition (3) it holds that $\mu(A(t_0,t_1)) \geqslant \mu(A(t_1-1)) \cdots \mu(A(t_0)) \geqslant \delta^{n^*}$. If $n^* \leqslant n^2 - n + 2$ we are ready. Otherwise we will need at least $n^2 - n + 2$ multiplications in $A(t_0, t_1)$ to reach $\mu(A(t_0,t_1)) < \delta^{n^2-n+2}$. We will show below that in each step where the positive minimum sinks we must lose one zero entry. Thus $\mu(A(t_0,t_1)) < \delta^{n^2-n+2}$ implies that $A(t_0)$ must have $n^2 - n + 2$ zeros more than $A(t_0, t_1)$ and thus cannot have a positive diagonal, a contradiction to condition (1).

In formal terms we have to show for two confidence matrices $A, B \in \mathbb{R}^{n\times n}_{\geqslant 0}$ fulfilling conditions (1) and (2) that it holds

$$\mu(AB) < \mu(B) \Longrightarrow \exists (i,j) \quad \text{such that } (AB)_{ij} > 0 \text{ and } B_{ij} = 0\,. \tag{A.1}$$

Due to property (1) it holds that all non-zero entries in B are also non-zero entries in AB. To prove (A.1) we assume that the zero patterns of AB and B are equal and derive $\mu(AB) \geqslant \mu(B)$.

Let $i,j \in \underline{n}$ be indices such that $(AB)_{ij} > 0$ (and $b_{ij} > 0$) we can conclude

$$\begin{aligned}(AB)_{ij} &= \sum_{k\in\underline{n} \text{ with } b_{kj}>0} a_{ik} b_{kj} \geqslant \left(\min_{k\in\underline{n} \text{ with } b_{kj}>0} b_{kj} \right) \left(\sum_{k\in\underline{n} \text{ with } b_{kj}>0} a_{ik} \right) \\ &\overset{(*)}{=} \min_{k \text{ with } b_{kj}>0} b_{kj} \geqslant \mu(B)\,.\end{aligned}$$

Equality (*) holds by $\sum_{k\in\underline{n} \text{ with } b_{kj}>0} a_{ik} = 1$ which holds by the following argument.

$$b_{kj} = 0 \Rightarrow (AB)_{kj} = 0 \Rightarrow \sum_{l=1}^{n} a_{kl} b_{lj} = 0 \overset{b_{ij}>0}{\Longrightarrow} a_{ki} = 0 \overset{(2)}{\Longrightarrow} a_{ki} = 0$$

A.3. Proof of Proposition 5

For $A \in \mathbb{R}^{n\times n}$ we can define the row-diameter $\mathrm{d}(A)$ as the maximum Euclidean distance of two arbitrary rows in A. It can be shown that multiplication from the left with a row-stochastic matrix $A \in \mathbb{R}^{n\times n}$ to a matrix $B \in \mathbb{R}^{n\times n}$ shrinks the row-diameter of B in this way

$$\mathrm{d}(AB) \leqslant \left(1 - \min_{i,j} \sum_{k=1}^{n} \min\{a_{ik}, a_{jk}\} \right) \mathrm{d}(B)\,. \tag{A.2}$$

Now we can conclude

$$\mathrm{d}(A(0,t+1)) \leqslant (1-\delta_t)\,\mathrm{d}(A(0,t)) \leqslant \mathrm{e}^{-\delta_t}\,\mathrm{d}(A(0,t)) \leqslant \mathrm{e}^{-\sum_{i=0}^{t}\delta_t}\,\mathrm{d}(A(0))\,.$$

Thus $\lim_{t\to\infty} \mathrm{d}(A(0,t)) = 0$ and this leads in our row-stochastic case to $\lim_{t\to\infty} A(0,t) = K$ consensus matrix.

Eq. (A.2) is a more dimensional version of the well known shrinking lemma, seen for example in Ref. [7]. For a proof see Ref. [6] (pp. 22–23, Satz 2.4.7).

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